Hooke's law

Learning Objectives

After completing this chapter, you should be able to:

- **Derive** Hooke's law from the fundamental response to volumetric and shear deformation
- **Relate** the bulk modulus K and shear modulus G to Young's modulus E and Poisson's ratio ν
- **Apply** Hooke's law to calculate stress from strain (and vice versa) in 3D problems

In order to compute the deformation of an elastic body, we need a constitutive relation (material law) to close the equations of elastostatic equilibrium,

$$\nabla \cdot \underline{\sigma} = 0 \quad \text{and} \quad \underline{\varepsilon}(\vec{r}) = \frac{1}{2}\left(\nabla\vec{u} + (\nabla\vec{u})^T\right). \tag{1}$$

Since we will be working in linear elasticity, the constitutive equation is a linear relationship between $\underline{\sigma}$ and $\underline{\varepsilon}$. Rather than simply stating this relationship, we will derive it from fundamental physical principles by considering how isotropic materials respond to two fundamental types of deformation: *volumetric* (hydrostatic) deformation and *shear* (deviatoric) deformation.

Volumetric and deviatoric decomposition

Decomposing stress and strain

Any symmetric tensor can be decomposed into a *spherical* (or *hydrostatic*) part and a *deviatoric* part. For the stress tensor:

$$\underline{\sigma} = \sigma_h \underline{1} + \underline{s}$$

where $\sigma_h = \frac{1}{3}\mathrm{tr}\,\underline{\sigma} = \frac{1}{3}\sigma_{kk}$ is the *hydrostatic stress* (equal to the mean of the principal stresses), $\underline{1}$ is the identity tensor, and $\underline{s} = \underline{\sigma} - \sigma_h\underline{1}$ is the *deviatoric stress tensor*. The deviatoric stress has zero trace by construction: $\mathrm{tr}\,\underline{s} = 0$.

Similarly, for the strain tensor:

$$\underline{\varepsilon} = \varepsilon_h \underline{1} + \underline{e}$$

where $\varepsilon_h = \frac{1}{3}\mathrm{tr}\,\underline{\varepsilon} = \frac{1}{3}\varepsilon_{kk}$ is the *volumetric strain* (related to relative volume change by $\Delta V/V = 3\varepsilon_h = \varepsilon_{kk}$), and $\underline{e} = \underline{\varepsilon} - \varepsilon_h \underline{1}$ is the *deviatoric strain tensor*.

Physical meaning

The hydrostatic part represents uniform compression or expansion—a change in volume without change in shape. Imagine a cube being uniformly squeezed from all sides: it becomes a smaller cube but remains cubic.

The deviatoric part represents pure distortion—a change in shape without change in volume. Imagine shearing a cube into a parallelepiped: it changes shape but its volume remains constant.

This decomposition is fundamental because, for isotropic materials, these two types of deformation are *independent* and governed by *different* elastic constants.

The two fundamental elastic moduli

Bulk modulus: Response to volumetric deformation

Consider a material subjected to pure hydrostatic stress, i.e., uniform pressure from all directions:

$$\underline{\sigma} = -p\,\underline{1} \quad \text{(pressure)}$$

where p is the pressure (positive in compression). The hydrostatic stress is $\sigma_h = -p$.

For an isotropic material, this produces a purely volumetric strain (uniform contraction):

$$\underline{\varepsilon} = \varepsilon_h\,\underline{1}$$

The *bulk modulus* K (also called the compression modulus) relates these:

$$\sigma_h = K \cdot 3\varepsilon_h = K \cdot \varepsilon_{kk} \tag{2}$$

or equivalently, $-p = K \cdot (\Delta V/V)$.

The bulk modulus measures the material's resistance to uniform compression. It has units of pressure (Pa). Typical values are:

- Steel: $K \approx 160$ GPa
- Aluminum: $K \approx 76$ GPa
- Rubber: $K \approx 2$ GPa
- Water: $K \approx 2.2$ GPa

A material with large K is difficult to compress volumetrically.

Shear modulus: Response to shear deformation

Now consider a material subjected to pure shear stress. For example, a simple shear in the x-z plane:

$$\underline{\sigma} = \begin{pmatrix} 0 & 0 & \tau \\ 0 & 0 & 0 \\ \tau & 0 & 0 \end{pmatrix}$$

This stress is purely deviatoric (zero trace) and produces a purely deviatoric strain:

$$\underline{\varepsilon} = \begin{pmatrix} 0 & 0 & \varepsilon_{xz} \\ 0 & 0 & 0 \\ \varepsilon_{xz} & 0 & 0 \end{pmatrix}$$

The *shear modulus* G (also denoted μ and called Lamé's second parameter) relates these:

$$\tau = 2G\varepsilon_{xz} = G\gamma_{xz} \tag{3}$$

where $\gamma_{xz} = 2\varepsilon_{xz}$ is the engineering shear strain.

More generally, the deviatoric stress is proportional to the deviatoric strain:

$$\underline{s} = 2G\,\underline{e} \tag{4}$$

The shear modulus measures the material's resistance to shape change. Typical values are:

- Steel: $G \approx 80$ GPa
- Aluminum: $G \approx 26$ GPa
- Rubber: $G \approx 0.001$ GPa

A material with large G is stiff in shear (difficult to distort).

Building the full constitutive relation

Combining volumetric and shear response

For an isotropic linear elastic material, we now combine the two fundamental responses. The stress can be written as:

$$\underline{\sigma} = \sigma_h\,\underline{1} + \underline{s} = K\varepsilon_{kk}\,\underline{1} + 2G\,\underline{e}$$

Substituting $\underline{e} = \underline{\varepsilon} - \frac{1}{3}\varepsilon_{kk}\,\underline{1}$:

$$\underline{\sigma} = K\varepsilon_{kk}\,\underline{1} + 2G\left(\underline{\varepsilon} - \frac{1}{3}\varepsilon_{kk}\,\underline{1}\right)$$

Rearranging:

$$\underline{\sigma} = \left(K - \frac{2G}{3}\right) \varepsilon_{kk}\, \underline{1} + 2G\, \underline{\varepsilon}$$

In component form:

$$\sigma_{ij} = \left(K - \frac{2G}{3}\right) \delta_{ij}\varepsilon_{kk} + 2G\varepsilon_{ij} \tag{5}$$

This is Hooke's law expressed in terms of K and G.

Lamé's first parameter

The combination $K - \frac{2G}{3}$ appears naturally and is given a name: *Lamé's first parameter*

$$\lambda = K - \frac{2G}{3} \tag{6}$$

With this definition, Hooke's law takes the compact form:

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} \tag{7}$$

where we use $\mu = G$ (Lamé's second parameter, same as the shear modulus).

This is the *forward* constitutive relation: given the strain, compute the stress.

The inverse relation

To find strain from stress, we invert the relationship. The inverse of Hooke's law is:

$$\varepsilon_{ij} = \frac{1}{2G}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}] \tag{8}$$

where E and ν are Young's modulus and Poisson's ratio, which we derive next.

Young's modulus and Poisson's ratio

Uniaxial tension experiment

While K and G have clear physical interpretations (resistance to compression and shear), the most common laboratory experiment is the *uniaxial tension test*: a bar is stretched in one direction (say x) with no stress in the transverse directions.

The stress state is:

$$\sigma_{xx} = \sigma, \quad \sigma_{yy} = \sigma_{zz} = 0, \quad \text{all shear stresses} = 0$$

Using Eq. (8), the strain state is:

$$\varepsilon_{xx} = \frac{\sigma}{E}, \quad \varepsilon_{yy} = \varepsilon_{zz} = -\frac{\nu\sigma}{E}$$

This defines:

- **Young's modulus** E: the ratio of axial stress to axial strain in uniaxial tension

$$E = \frac{\sigma}{\varepsilon_{xx}}$$

- **Poisson's ratio** ν: the negative ratio of transverse strain to axial strain

$$\nu = -\frac{\varepsilon_{yy}}{\varepsilon_{xx}} = -\frac{\varepsilon_{zz}}{\varepsilon_{xx}}$$

Relating E, nu to K, G

We can derive the relationships by considering the uniaxial stress state. With $\sigma_{xx} = \sigma$ and all other stresses zero:

- Hydrostatic stress: $\sigma_h = \frac{\sigma}{3}$
- Volumetric strain: $\varepsilon_{kk} = \frac{\sigma_h}{K} = \frac{\sigma}{3K}$

For the axial strain, using the deviatoric part:

$$\varepsilon_{xx} = \varepsilon_h + e_{xx} = \frac{\varepsilon_{kk}}{3} + \frac{s_{xx}}{2G}$$

The deviatoric stress is $s_{xx} = \sigma_{xx} - \sigma_h = \sigma - \frac{\sigma}{3} = \frac{2\sigma}{3}$.

Therefore:

$$\varepsilon_{xx} = \frac{\sigma}{9K} + \frac{2\sigma}{6G} = \frac{\sigma}{9K} + \frac{\sigma}{3G} = \sigma\left(\frac{1}{9K} + \frac{1}{3G}\right)$$

Since $E = \sigma/\varepsilon_{xx}$:

$$\boxed{E = \frac{9KG}{3K + G}} \tag{9}$$

For the transverse strain:

$$\varepsilon_{yy} = \varepsilon_h + e_{yy} = \frac{\sigma}{9K} + \frac{s_{yy}}{2G}$$

The deviatoric stress is $s_{yy} = 0 - \frac{\sigma}{3} = -\frac{\sigma}{3}$.

Therefore:

$$\varepsilon_{yy} = \frac{\sigma}{9K} - \frac{\sigma}{6G}$$

The Poisson ratio is:

$$\nu = -\frac{\varepsilon_{yy}}{\varepsilon_{xx}} = \frac{-\frac{1}{9K} + \frac{1}{6G}}{\frac{1}{9K} + \frac{1}{3G}} = \frac{\frac{3K-2G}{18KG}}{\frac{3K+G}{9KG}} = \frac{3K-2G}{2(3K+G)}$$

$$\boxed{\nu = \frac{3K-2G}{2(3K+G)}} \tag{10}$$

Inverse relations

We can also express K and G in terms of E and ν:

$$K = \frac{E}{3(1-2\nu)} \tag{11}$$

$$G = \frac{E}{2(1+\nu)} \tag{12}$$

And Lamé's first parameter:

$$\lambda = K - \frac{2G}{3} = \frac{E\nu}{(1+\nu)(1-2\nu)} \tag{13}$$

! Poisson's ratio $\nu = 0.5$ means volume conservation

From Eq. (11), we see that:

$$K = \frac{E}{3(1-2\nu)}$$

As $\nu \to 0.5$, the bulk modulus $K \to \infty$. This means the material becomes *incompressible*: it cannot change its volume.
Let's verify this directly. The volumetric strain is:

$$\varepsilon_{kk} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

For uniaxial tension with $\varepsilon_{xx} = \sigma/E$ and $\varepsilon_{yy} = \varepsilon_{zz} = -\nu\sigma/E$:

$$\varepsilon_{kk} = \frac{\sigma}{E} - 2\frac{\nu\sigma}{E} = \frac{\sigma}{E}(1-2\nu)$$

When $\nu = 0.5$:

$$\varepsilon_{kk} = 0 \quad \Rightarrow \quad \frac{\Delta V}{V} = 0$$

The material stretches in the axial direction but contracts in the transverse directions in exactly the right proportion to keep the volume constant.
Physical examples:

- **Rubber** has $\nu \approx 0.5$ (nearly incompressible)
- **Water** and other liquids have $\nu = 0.5$ exactly (truly incompressible)
- **Metals** typically have $\nu \approx 0.3$ (compressible)

The physical bounds on Poisson's ratio are $-1 < \nu < 0.5$. The upper limit $\nu = 0.5$ corresponds to incompressibility. Materials with $\nu < 0$ (auxetic materials) expand transversely when stretched—these are typically architectured materials like certain foams.

General form of Hooke's law

The stiffness tensor

The most general form of the linear relationship between stress and strain is:

$$\underline{\sigma} = \underline{\underline{C}} : \underline{\varepsilon}, \quad \text{or} \quad \sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{14}$$

This is *Hooke's law*. The quantity $\underline{\underline{C}}$ is a fourth-order tensor of elastic constants called the *stiffness tensor*. For a general anisotropic material, it contains at most 21 independent elastic constants.

To see that there are only 21 independent coefficients, we use *Voigt notation*, which removes the symmetric entries from $\underline{\sigma}$ and $\underline{\varepsilon}$ and expresses them as 6-vectors:

$$\vec{\sigma} = (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{xz}, \sigma_{xy}) \tag{15}$$

and

$$\vec{\varepsilon} = (\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, 2\varepsilon_{yz}, 2\varepsilon_{xz}, 2\varepsilon_{xy}) \tag{16}$$

Then $\vec{\sigma} = \underline{C} \cdot \vec{\varepsilon}$ where $\underline{C}$ is a 6×6 symmetric matrix called the *stiffness matrix*. The symmetry reduces the 36 components to 21 independent constants.

Notation for shear components

The off-diagonal components of $\underline{\sigma}$ are often denoted by $\tau_{xy} \equiv \sigma_{xy}$, $\tau_{xz} \equiv \sigma_{xz}$, and $\tau_{yz} \equiv \sigma_{yz}$. The off-diagonal components of $\underline{\varepsilon}$ are often written as *engineering shear strains*: $\gamma_{xy} \equiv 2\varepsilon_{xy}$, $\gamma_{xz} \equiv 2\varepsilon_{xz}$, and $\gamma_{yz} \equiv 2\varepsilon_{yz}$.

Note: The γ's contain a factor of 2 but the τ's do not. This factor ensures that $\vec{\sigma} = \underline{C} \cdot \vec{\varepsilon}$ and $\underline{\sigma} = \underline{\underline{C}} : \underline{\varepsilon}$ are equivalent.

Stiffness matrix for isotropic materials

For isotropic materials, the stiffness matrix in Voigt notation is:

$$\underline{C} = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix}$$

This confirms that isotropic materials have only 2 independent elastic constants (λ and μ), compared to 21 for general anisotropic materials.

Elastic energy

We can also write down a *free energy functional* (often called hyperelastic energy density), which is quadratic in the strain $\underline{\varepsilon}$:

$$W = \frac{1}{2}\lambda\varepsilon_{ii}^2 + \mu\varepsilon_{ij}^2 \tag{17}$$

Using $\sigma_{ij} = \partial W/\partial\varepsilon_{ij}$ recovers Hooke's law (Eq. (7)).

In terms of K and G, the energy can be written as:

$$W = \frac{1}{2}K\varepsilon_{kk}^2 + G\, e_{ij}e_{ij} \tag{18}$$

This form clearly separates the volumetric energy (first term, proportional to K) from the shear/distortional energy (second term, proportional to G). For the energy to be positive definite (so that deformation costs energy), we require $K > 0$ and $G > 0$.

These constraints limit the Poisson ratio to the range $-1 < \nu < 1/2$.

Summary of elastic constants

The following table summarizes the relationships between pairs of elastic constants:

Given	E	ν	K	G	λ
K, G	$\frac{9KG}{3K+G}$	$\frac{3K-2G}{2(3K+G)}$	K	G	$K - \frac{2G}{3}$
E, ν	E	ν	$\frac{E}{3(1-2\nu)}$	$\frac{E}{2(1+\nu)}$	$\frac{E\nu}{(1+\nu)(1-2\nu)}$

Given	E	ν	K	G	λ
λ, μ	$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\frac{\lambda}{2(\lambda+\mu)}$	$\lambda + \frac{2\mu}{3}$	μ	λ

Example: Uniaxial tension test

Tip

Consider a rectangular block of isotropic elastic material subjected to uniaxial tension. The block has Young's modulus $E = 200$ GPa and Poisson's ratio $\nu = 0.3$. An axial stress $\sigma_{xx} = 100$ MPa is applied in the x-direction with all other stresses zero: $\sigma_{yy} = \sigma_{zz} = \tau_{yz} = \tau_{xz} = \tau_{xy} = 0$.
Find:

- The axial strain ε_{xx}
- The transverse strains ε_{yy} and ε_{zz}
- The volumetric strain ε_h

Solution:
This is the simplest and most fundamental loading case in elasticity: a bar being stretched in one direction. We want to find how much the bar elongates in the loading direction and how much it contracts in the perpendicular directions.
We use the inverse constitutive relation (Eq. (8)) that gives strain in terms of stress:

$$\varepsilon_{ij} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}]$$

This formula contains the trace of the stress tensor, σ_{kk}, which is the sum of the diagonal components. We calculate this first:

$$\sigma_{kk} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 100 + 0 + 0 = 100 \text{ MPa}$$

Now we apply the formula to find each strain component. For the axial strain ε_{xx}, we set $i = j = x$. The Kronecker delta $\delta_{xx} = 1$, so:

$$\varepsilon_{xx} = \frac{1}{E}[(1+\nu)\sigma_{xx} - \nu\sigma_{kk}]$$

Substituting the values ($E = 200$ GPa $= 200 \times 10^3$ MPa, $\nu = 0.3$, $\sigma_{xx} = 100$ MPa, $\sigma_{kk} = 100$ MPa):

$$\varepsilon_{xx} = \frac{1}{200 \times 10^3}[(1+0.3)(100) - (0.3)(100)] = \frac{1}{200 \times 10^3}[130 - 30] = \frac{100}{200 \times 10^3}$$

$$= 0.5 \times 10^{-3} = 5 \times 10^{-4}$$

This means the bar elongates by 0.5 mm for every meter of original length (or equivalently, 0.05%).
For the transverse strain ε_{yy}, we set $i = j = y$. Again $\delta_{yy} = 1$, and now $\sigma_{yy} = 0$:

$$\varepsilon_{yy} = \frac{1}{E}[(1+\nu)\sigma_{yy} - \nu\sigma_{kk}] = \frac{1}{200 \times 10^3}[(1.3)(0) - (0.3)(100)]$$

$$= \frac{1}{200 \times 10^3}[-30] = -0.15 \times 10^{-3} = -1.5 \times 10^{-4}$$

The negative sign indicates contraction: the bar gets thinner as it stretches. By symmetry of the loading and material (isotropic), the z-direction behaves the same as the y-direction: $\varepsilon_{zz} = -0.15 \times 10^{-3}$.
The volumetric strain ε_h (also called dilatation) is the sum of the normal strains:

$$\varepsilon_h = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = (0.5 - 0.15 - 0.15) \times 10^{-3} = 0.2 \times 10^{-3}$$

We can verify this using the alternative formula from the text. The hydrostatic stress is $\sigma_h = \sigma_{kk}/3 = 100/3$ MPa, and:

$$\varepsilon_h = \frac{1}{E}(1-2\nu)\sigma_{kk} = \frac{1}{200 \times 10^3}(1 - 0.6)(100) = \frac{40}{200 \times 10^3} = 0.2 \times 10^{-3}$$

This confirms our calculation. The positive volumetric strain means the material increases in volume when stretched—the elongation in the x-direction is not fully compensated by the contractions in the y and z directions. This is true for all materials with $\nu < 0.5$. An incompressible material ($\nu = 0.5$) would have $\varepsilon_h = 0$.
Finally, we can verify that the Poisson's ratio is recovered correctly. The ratio of transverse to axial strain should equal $-\nu$:

$$\frac{\varepsilon_{yy}}{\varepsilon_{xx}} = \frac{-0.15 \times 10^{-3}}{0.5 \times 10^{-3}} = -0.3 = -\nu \quad \checkmark$$

i Chapter Summary

This chapter derived Hooke's law for linear elastic materials from fundamental principles:

- **Volumetric response**: Hydrostatic stress σ_h relates to volumetric strain via bulk modulus K: $\sigma_h = K\varepsilon_{kk}$
- **Shear response**: Deviatoric stress relates to deviatoric strain via shear modulus G: $\underline{s} = 2G\,\underline{e}$
- **Combined law**: $\sigma_{ij} = (K - \frac{2G}{3})\delta_{ij}\varepsilon_{kk} + 2G\varepsilon_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$
- **Young's modulus and Poisson ratio**: $E = \frac{9KG}{3K+G}$ and $\nu = \frac{3K-2G}{2(3K+G)}$
- **Inverse relations**: $K = \frac{E}{3(1-2\nu)}$ and $G = \frac{E}{2(1+\nu)}$

- **Incompressibility**: $\nu = 0.5$ implies $K \to \infty$ and zero volumetric strain
- **General form**: $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ with 21 constants for anisotropic, 2 for isotropic materials
- **Inverse relation**: $\varepsilon_{ij} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}]$

Hooke's law is the constitutive relation that closes the elasticity equations and enables calculation of deformations.