# Strain and displacement

Learning Objectives

After completing this chapter, you should be able to:

- **Define** the displacement field $\vec{u}(\vec{r})$ as the change in position during deformation
- **Derive** the strain tensor $\underline{\varepsilon} = \frac{1}{2}(\nabla\vec{u} + (\nabla\vec{u})^T)$ from displacement gradients
- **Interpret** normal strains as stretching and shear strains as angular distortion

In the previous chapter, we introduced the stress tensor and derived the equilibrium equations that govern forces inside a solid body. However, equilibrium alone does not tell us how a body deforms under load. To connect forces to deformation, we need to introduce the concepts of *displacement* and *strain*. Together with Hooke's law (which we will discuss in the next chapter), these concepts complete the set of equations needed to solve problems in linear elasticity.

## Displacement

When a solid body is subjected to external forces, its material points move from their original positions to new positions. The *displacement field* $\vec{u}(\vec{r})$ describes this motion mathematically. Given a material point that was originally located at position $\vec{r}$ in the undeformed (reference) configuration, its new position after deformation is

$$\vec{r}' = \vec{r} + \vec{u}(\vec{r}).$$

The displacement $\vec{u}$ is therefore the vector pointing from the original position to the deformed position. Note that the displacement field is defined as a function of the *reference* position $\vec{r}$, not the deformed position $\vec{r}'$. This choice, called the *Lagrangian* or *material* description, is standard in solid mechanics because it allows us to track what happens to specific material points.

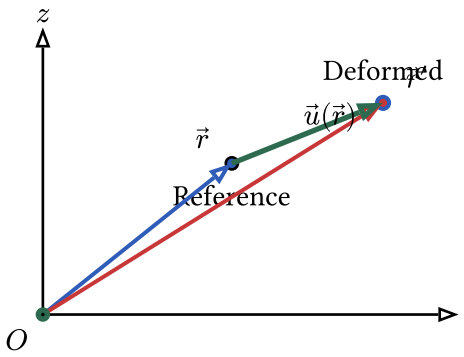



Figure 1: Displacement of a material point from its reference position $\vec{r}$ to its deformed position $\vec{r}'$. The displacement vector $\vec{u}(\vec{r})$ connects the two positions.

The displacement field $\vec{u}(\vec{r})$ has three components in three-dimensional space,

$$\vec{u} = \begin{pmatrix} u_x(x,y,z) \\ u_y(x,y,z) \\ u_z(x,y,z) \end{pmatrix},$$

where each component can vary with position throughout the body. A uniform displacement $\vec{u} = \text{const}$ corresponds to a rigid-body translation: all points move by the same amount, and the body does not deform. More interesting cases arise when the displacement varies from point to point, which leads to the concept of strain.

## From displacement to strain

### Why the displacement gradient matters

To understand deformation, we need to examine how displacement varies spatially. Consider two neighboring material points, one at position $\vec{r}$ and another at $\vec{r} + \mathrm{d}\vec{r}$, where $\mathrm{d}\vec{r}$ is an infinitesimally small vector connecting them. In the reference configuration, these points are separated by the distance $|\mathrm{d}\vec{r}|$. After deformation, the first point moves to $\vec{r} + \vec{u}(\vec{r})$ and the second to $\vec{r} + \mathrm{d}\vec{r} + \vec{u}(\vec{r} + \mathrm{d}\vec{r})$. The vector connecting them in the deformed configuration is

$$\mathrm{d}\vec{r}' = \mathrm{d}\vec{r} + \vec{u}(\vec{r} + \mathrm{d}\vec{r}) - \vec{u}(\vec{r}).$$

Using a Taylor expansion and keeping only first-order terms (which is valid for small displacements), we obtain

$$\vec{u}(\vec{r} + \mathrm{d}\vec{r}) - \vec{u}(\vec{r}) \approx (\nabla \vec{u}) \cdot \mathrm{d}\vec{r},$$

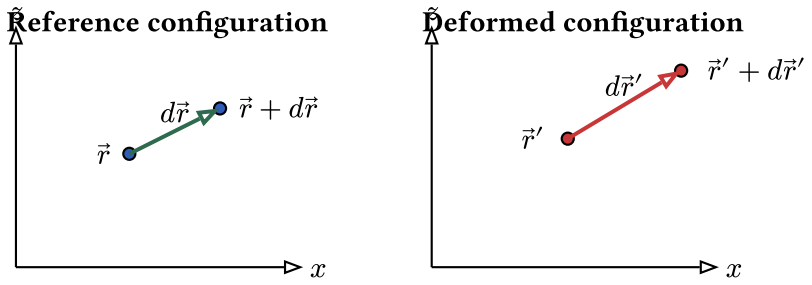



Figure 2: Two neighboring points separated by $\mathrm{d}\vec{r}$ in the reference configuration. After deformation, their separation becomes $\mathrm{d}\vec{r}'$. The change in separation depends on the displacement gradient.

where $\nabla\vec{u}$ is the *displacement gradient tensor*. This second-rank tensor contains all the information about how the displacement field varies in space. Its components are the partial derivatives

$$[\nabla\vec{u}]_{ij} = \frac{\partial u_i}{\partial r_j} \equiv u_{i,j},$$

where we have introduced the comma notation $u_{i,j}$ as shorthand for partial differentiation. Written as a matrix, the displacement gradient is

$$\nabla\vec{u} = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{pmatrix}.$$

It is important to distinguish the displacement gradient $\nabla\vec{u}$ (a tensor) from the divergence $\nabla\cdot\vec{u}$ (a scalar). The gradient tells us how the vector field changes in different directions, while the divergence only tells us whether the field is expanding or contracting at a point.

## Decomposing the displacement gradient

The displacement gradient $\nabla\vec{u}$ is a general second-rank tensor that need not be symmetric. Any such tensor can be uniquely decomposed into a symmetric part and an antisymmetric part,

$$\nabla\vec{u} = \underbrace{\frac{1}{2}\left(\nabla\vec{u} + (\nabla\vec{u})^T\right)}_{\underline{\varepsilon}} + \underbrace{\frac{1}{2}\left(\nabla\vec{u} - (\nabla\vec{u})^T\right)}_{\underline{\omega}}.$$

The symmetric part $\underline{\varepsilon}$ is called the *strain tensor*, and the antisymmetric part $\underline{\omega}$ is called the *rotation tensor*. This decomposition has profound physical significance: the strain tensor describes pure deformation (changes in shape and size), while the rotation tensor describes rigid-body rotation (changes in orientation without deformation).

## The strain tensor

### Definition and components

The *strain tensor* is defined as the symmetric part of the displacement gradient,

$$\underline{\varepsilon}(\vec{r}) = \frac{1}{2}\left(\nabla\vec{u} + (\nabla\vec{u})^T\right). \tag{1}$$

In component notation, the strain tensor elements are

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i}\right). \tag{2}$$

Note that this expression does not contain an implicit sum because there are no repeated indices. The symmetrization ensures that $\varepsilon_{ij} = \varepsilon_{ji}$, so the strain tensor has only six independent components rather than nine.

Written explicitly as a matrix, the strain tensor is

$$\underline{\varepsilon} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{pmatrix}$$

where the diagonal components are

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}, \tag{3}$$

and the off-diagonal components are

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \tag{4}$$

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right), \tag{5}$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right). \tag{6}$$

The strain tensor is dimensionless because it is the ratio of a displacement (length) to a reference length. Typical values in engineering applications range from $10^{-6}$ (microstrains) to $10^{-2}$ (a few percent strain).

## Geometric interpretation of normal strains

The diagonal components of the strain tensor, called *normal strains*, describe stretching or compression along the coordinate axes. Consider a small line element of length $\mathrm{d}x$ oriented along the $x$-axis in the reference configuration. After deformation, the $x$-component of this element becomes

$$\mathrm{d}x' = \mathrm{d}x + u_x(x + \mathrm{d}x, y, z) - u_x(x, y, z) \approx \mathrm{d}x + \frac{\partial u_x}{\partial x}\mathrm{d}x = (1 + \varepsilon_{xx})\mathrm{d}x.$$

The normal strain $\varepsilon_{xx}$ is therefore the relative change in length of a line element originally aligned with the $x$-axis,

$$\varepsilon_{xx} = \frac{\mathrm{d}x' - \mathrm{d}x}{\mathrm{d}x} = \frac{\Delta L}{L}.$$

Positive normal strain indicates elongation (tension), while negative normal strain indicates contraction (compression). The same interpretation applies to $\varepsilon_{yy}$ and $\varepsilon_{zz}$ for line elements along the $y$- and $z$-axes.

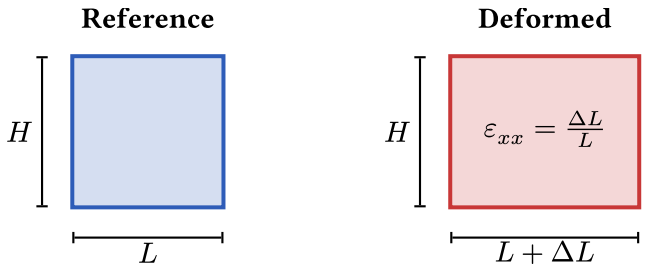



Figure 3: Normal strain $\varepsilon_{xx}$ represents the relative elongation of a line element along the $x$-axis. A square element becomes a rectangle with the same height but different width.

## Geometric interpretation of shear strains

The off-diagonal components, called *shear strains*, describe angular distortion—changes in the angles between line elements that were originally perpendicular. Consider two line elements, one along the $x$-axis and one along the $y$-axis, meeting at a right angle in the reference configuration. After deformation, these elements are no longer perpendicular; the angle between them changes by an amount related to the shear strain.

To see this, consider the displacement of a point at $(x + \mathrm{d}x, y)$ relative to a point at $(x, y)$. The $y$-component of this relative displacement is

$$\frac{\partial u_y}{\partial x}\mathrm{d}x,$$

which represents a rotation of the originally horizontal line element toward the $y$-axis. Similarly, a point at $(x, y+\mathrm{d}y)$ experiences a relative $x$-displacement of $(\partial u_x/\partial y)\mathrm{d}y$, rotating the originally vertical line element toward the $x$-axis.




Figure 4: Shear strain $\varepsilon_{xy}$ represents the angular distortion of a rectangular element. The angles $\alpha$ and $\beta$ measure the rotation of the horizontal and vertical edges, respectively. The shear strain is half of the total angle change: $\varepsilon_{xy} = (\alpha + \beta)/2$.

If we denote the rotation of the horizontal edge by angle $\alpha = \partial u_y/\partial x$ and the rotation of the vertical edge by angle $\beta = \partial u_x/\partial y$ (both measured in radians and assumed small), then the total change in angle between the two edges is $\alpha+\beta$. The shear strain component $\varepsilon_{xy}$ is defined as half of this total angle change,

$$\varepsilon_{xy} = \frac{1}{2}(\alpha + \beta) = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right).$$

The factor of $\frac{1}{2}$ in the definition ensures that the strain tensor transforms correctly under coordinate rotations (i.e., as a proper second-rank tensor). In engineering practice, one often encounters the *engineering shear strain* $\gamma_{xy} = 2\varepsilon_{xy}$, which equals the total angle change directly. When reading literature or handbooks, it is important to check which definition is being used.

## The rotation tensor

The antisymmetric part of the displacement gradient,

$$\underline{\omega} = \frac{1}{2}\left(\nabla\vec{u} - (\nabla\vec{u})^T\right),$$

describes rigid-body rotation. An antisymmetric tensor satisfies $\omega_{ij} = -\omega_{ji}$, which means its diagonal elements are zero and it has only three independent components. These can be collected into a vector $\vec{\omega}$ called the *rotation vector*,

$$\omega_x = \omega_{zy} = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right), \quad \text{etc.}$$

This rotation vector is related to the curl of the displacement field by $\vec{\omega} = \frac{1}{2}\nabla \times \vec{u}$.

For small rotations, the rotation tensor $\underline{\omega}$ represents the local rigid-body rotation of material elements without any change in their shape or size. Pure rotation (without strain) occurs when $\underline{\varepsilon} = 0$ but $\underline{\omega} \neq 0$. Conversely, pure deformation (without rotation) occurs when $\underline{\omega} = 0$ but $\underline{\varepsilon} \neq 0$.

## Physical interpretation as a linear transformation

The strain tensor $\underline{\varepsilon}$ is a linear transformation that maps a position vector to a displacement vector. For a material point at position $\vec{r}$ in a uniformly strained body, the displacement is

$$\vec{u} = \underline{\varepsilon} \cdot \vec{r}.$$

The deformed position is then

$$\vec{r}\,' = \vec{r} + \vec{u} = (\underline{1} + \underline{\varepsilon}) \cdot \vec{r},$$

where $\underline{1}$ is the identity tensor. This shows that the deformation gradient tensor $\underline{F} = \underline{1} + \underline{\varepsilon}$ transforms reference positions into deformed positions.

Since the strain tensor is symmetric, it can be diagonalized by a rotation of the coordinate system. In the principal coordinate system, the strain tensor takes the form

$$\underline{\varepsilon}' = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix},$$

where $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$ are the *principal strains*. In this coordinate system, there are no shear strains—only normal strains along the principal directions. A cube aligned with the principal axes deforms into a rectangular parallelepiped with edges stretched or compressed by factors $(1 + \varepsilon_1)$, $(1 + \varepsilon_2)$, and $(1 + \varepsilon_3)$.

## Volumetric and deviatoric strain

Just as we decomposed the stress tensor into hydrostatic and deviatoric parts, we can decompose the strain tensor similarly. The *volumetric strain* (or *dilatation*) is defined as the trace of the strain tensor,

$$\varepsilon_{\text{vol}} = \operatorname{tr} \underline{\varepsilon} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \nabla \cdot \vec{u}.$$

This quantity represents the relative change in volume of an infinitesimal element. To see this, consider a small cube with edges $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$ that deforms into a rectangular parallelepiped with edges $(1 + \varepsilon_{xx})\mathrm{d}x$, $(1 + \varepsilon_{yy})\mathrm{d}y$, $(1 + \varepsilon_{zz})\mathrm{d}z$. The ratio of deformed to reference volume is

$$\frac{V'}{V} = (1 + \varepsilon_{xx})(1 + \varepsilon_{yy})(1 + \varepsilon_{zz}) \approx 1 + \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = 1 + \varepsilon_{\text{vol}},$$

where we have neglected products of small strains. Thus $\varepsilon_{\text{vol}} = (V' - V)/V = \Delta V/V$.

The *deviatoric strain tensor* is defined as

$$\underline{e} = \underline{\varepsilon} - \frac{1}{3}\varepsilon_{\text{vol}}\,\underline{1},$$

which has zero trace by construction. The deviatoric strain represents shape change at constant volume (distortion), while the volumetric part represents volume change without shape change (dilatation). This decomposition will prove useful when we discuss Hooke's law, where the volumetric and deviatoric responses are governed by different elastic constants.

## Example: Uniform extension

Consider a bar being stretched uniformly along the $x$-axis. If the bar has length $L$ and is stretched to length $L + \Delta L$, the displacement field is

$$u_x = \frac{\Delta L}{L}x, \quad u_y = -\nu\frac{\Delta L}{L}y, \quad u_z = -\nu\frac{\Delta L}{L}z,$$

where $\nu$ is Poisson's ratio (which we will discuss in the next chapter). The strain tensor is

$$\underline{\varepsilon} = \frac{\Delta L}{L}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & 0 & -\nu \end{pmatrix}.$$

The axial strain is $\varepsilon_{xx} = \Delta L/L$, and the transverse strains are $\varepsilon_{yy} = \varepsilon_{zz} = -\nu\Delta L/L$. The volumetric strain is

$$\varepsilon_{\text{vol}} = (1 - 2\nu)\frac{\Delta L}{L}.$$

For most materials $0 < \nu < 0.5$, so the volumetric strain is positive (the bar expands in volume when stretched). For an incompressible material ($\nu = 0.5$), the volumetric strain is zero—the bar gets longer and thinner such that its volume remains constant.

**Chapter Summary**

This chapter introduced the concepts of displacement and strain that describe the kinematics of deformable bodies.
The *displacement field* $\vec{u}(\vec{r}) = \vec{r}' - \vec{r}$ describes how each material point moves from its reference position $\vec{r}$ to its deformed position $\vec{r}'$. When displacement varies spatially, the body deforms.
The *displacement gradient* $\nabla\vec{u}$ is a second-rank tensor whose components $[\nabla\vec{u}]_{ij} = \partial u_i/\partial r_j$ describe how the displacement field varies in space. This tensor can be decomposed into symmetric and antisymmetric parts.

The *strain tensor* $\underline{\varepsilon} = \frac{1}{2}(\nabla\vec{u} + (\nabla\vec{u})^T)$ is the symmetric part of the displacement gradient. Its components $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ have clear geometric interpretations. Normal strains ($\varepsilon_{xx}$, $\varepsilon_{yy}$, $\varepsilon_{zz}$) measure relative elongation along the coordinate axes. Shear strains ($\varepsilon_{xy}$, etc.) measure angular distortion between originally perpendicular lines. The engineering shear strain $\gamma_{ij} = 2\varepsilon_{ij}$ equals the total angle change.
The *rotation tensor* $\underline{\omega} = \frac{1}{2}(\nabla\vec{u} - (\nabla\vec{u})^T)$ is the antisymmetric part and describes local rigid-body rotation without deformation.
The *volumetric strain* $\varepsilon_{\text{vol}} = \operatorname{tr}\underline{\varepsilon} = \nabla \cdot \vec{u}$ represents the relative change in volume of an infinitesimal element. The *deviatoric strain* $\underline{e} = \underline{\varepsilon} - \frac{1}{3}\varepsilon_{\text{vol}}\,\underline{1}$ represents shape change at constant volume.
Together with the equilibrium equation and Hooke's law, the strain-displacement relation completes the set of equations needed to solve problems in linear elasticity.