16. Buckling

Learning Objectives

After completing this chapter, you should be able to:

- **Explain** buckling as a loss of stability of an equilibrium
- **Derive** the buckling equation $EIw'''' + Nw'' = 0$ from the equilibrium of the deformed beam element
- **Calculate** the Euler critical load $N_{\text{cr}} = \pi^2 EI/L_0^2$ with the effective (buckling) length L_0 for the four Euler cases

So far, equilibrium was always written in the *undeformed* configuration: we assumed that the displacements are so small that they do not change the geometry on which the forces act. This is the essence of linear (first-order) theory, and it is why all our results were proportional to the load. For a slender beam under **compression** this assumption fails. A small lateral deflection gives the compressive force a lever arm, the lever arm creates a bending moment, the moment increases the deflection — and above a critical load the straight configuration is no longer stable. This chapter studies this loss of stability, called **buckling**.

Stability of an equilibrium

An equilibrium can be stable or unstable. A ball at the bottom of a valley returns after a small push (**stable**); a ball on top of a hill rolls away (**unstable**); a ball on a flat table stays wherever it is pushed (**neutral**, or indifferent). A straight column under a compressive force N is in equilibrium for every N. For small N this equilibrium is stable: a small lateral push produces a deflection that disappears when the push is removed. At the **critical load** N_{cr} the straight configuration becomes neutral, and a deflected shape is also in equilibrium. Above N_{cr} the straight configuration is unstable and the column buckles.

To find N_{cr} we therefore look for loads at which a **non-zero** deflection $w(x)$ is an equilibrium configuration.

Equilibrium in the deformed configuration

Consider a straight beam of length L with pinned ends, loaded at its ends by a compressive force N along the original axis. In this chapter N denotes the **compressive** axial force, $N > 0$ in compression (the opposite sign of the normal force in Chapter 4). Suppose the beam has a small lateral deflection $w(x)$ (positive downward). We cut the beam at x and write the moment equilibrium of the left part *in the deformed configuration*. The force N acts along the line through the two hinges, $z = 0$, while the cross-section at x has moved down by $w(x)$. The compressive force therefore acts at a distance w *above* the centroid of the cross-section. It compresses the top and stretches the bottom of the beam, which is a positive bending moment,

$$M(x) = N\, w(x).$$

With the Euler–Bernoulli relation $M = -EI_y\, w''$ (Chapter 7) we obtain

$$EI_y\, w'' + N\, w = 0.$$

Differentiating twice, and adding a transverse load $q(x)$ for generality, gives the form that holds for any support conditions,

$$\boxed{EI_y\, w'''' + N\, w'' = q(x).}$$

Compared with Chapter 7, the new term Nw'' is the transverse component of the compressive force on the deformed (inclined) beam element. It is called the second-order or **P–delta effect**. For a tensile force ($N < 0$) the term stiffens the beam; for a compressive force it softens it.

Buckling is a sudden lateral deformation of a slender structural member that occurs under compressive load, resulting in a loss of load-carrying capacity. Unlike simple compression, where the member shortens uniformly, buckling involves a sudden transition to a bending mode when a critical load is exceeded. This section examines the buckling of slender beams under axial compression.

Pin-pin column: problem setup

Consider a slender beam of length L with uniform cross-section under uniform axial compression N (positive for compression). The beam is pinned at both ends (pin-pin boundary conditions). When the axial force exceeds a critical value, the beam becomes unstable and deforms laterally.

The governing equation combines the Euler-Bernoulli beam equation with the effect of axial compression. When the beam bends laterally by an amount $w(x)$, the axial force creates an additional bending moment (called the P-delta effect):

$$EI_y w_{,xxxx} + N w_{,xx} = 0 \tag{1}$$

This is the **buckling equation** for a beam under axial compression with no external distributed load.

Boundary conditions

For a beam with pin-pin (simply supported) boundary conditions at both ends:

$$w(0) = 0 \tag{2}$$
$$w(L) = 0 \tag{3}$$
$$w_{,xx}(0) = 0 \tag{4}$$
$$w_{,xx}(L) = 0 \tag{5}$$

The first two conditions enforce zero displacement at the supports, while the last two enforce zero bending moment (free rotation).

Solution of the buckling equation

Let us rewrite the buckling equation in the form:

$$w_{,xxxx} + \frac{N}{EI_y} w_{,xx} = 0 \tag{6}$$

Define $\beta^2 = \frac{N}{EI_y}$. Then:

$$w_{,xxxx} + \beta^2 w_{,xx} = 0 \tag{7}$$

This is a fourth-order linear ODE. Let $v(x) = w_{,xx}$, giving:

$$v_{,xx} + \beta^2 v = 0 \tag{8}$$

The general solution is:

$$v(x) = A\cos(\beta x) + B\sin(\beta x) \tag{9}$$

Integrating twice to find $w(x)$:

$$w(x) = C_1 + C_2 x + \frac{A}{\beta^2}\cos(\beta x) + \frac{B}{\beta^2}\sin(\beta x) \tag{10}$$

Applying the boundary condition $w(0) = 0$:

$$C_1 + \frac{A}{\beta^2} = 0 \implies C_1 = -\frac{A}{\beta^2} \tag{11}$$

Applying the boundary condition $w_{,xx}(0) = 0$:

$$-A\beta^2 = 0 \implies A = 0 \tag{12}$$

This gives $C_1 = 0$. The solution reduces to:

$$w(x) = C_2 x + \frac{B}{\beta^2}\sin(\beta x) \tag{13}$$

Applying the boundary condition $w(L) = 0$:

$$C_2 L + \frac{B}{\beta^2}\sin(\beta L) = 0 \tag{14}$$

Applying the boundary condition $w_{,xx}(L) = 0$:

$$w_{,xx}(x) = -B\beta^2 \sin(\beta x) \tag{15}$$

At $x = L$:

$$-B\beta^2 \sin(\beta L) = 0 \tag{16}$$

This equation is satisfied if either $B = 0$ (which, with $w(L) = 0$, also gives $C_2 = 0$ and hence the trivial solution $w(x) = 0$, the straight beam) or $\sin(\beta L) = 0$. In the second case $w(L) = 0$ gives $C_2 = 0$, and the buckled shape is

$$w(x) = \frac{B}{\beta^2}\sin(\beta x), \tag{17}$$

a half sine wave (for the lowest mode) whose amplitude is *not determined*: at the critical load, deflected configurations of any (small) amplitude are in equilibrium — the neutral equilibrium of the previous section.

Euler buckling formula

The non-trivial buckling solutions occur when:

$$\sin(\beta L) = 0 \implies \beta L = n\pi, \quad n = 1, 2, 3, ... \tag{18}$$

Therefore:

$$\beta = \frac{n\pi}{L} \quad \text{and} \quad N = \beta^2 E I_y = \frac{n^2 \pi^2 E I_y}{L^2} \tag{19}$$

The **critical buckling load** (Euler buckling load) occurs at the lowest mode, $n = 1$:

$$\boxed{N_{\text{cr}} = \frac{\pi^2 E I_y}{L^2}} \tag{20}$$

This is the **Euler buckling formula** for a pin-pin beam. The buckling load is proportional to the flexural rigidity EI_y and inversely proportional to the square of the length L^2.

Physical interpretation

1. **Slenderness ratio**: The critical load depends on L^2, meaning slender beams buckle at much lower loads than stocky beams. With the radius of gyration $r = \sqrt{I_y/A}$ and the slenderness ratio $\lambda = L/r$, the critical *stress* is $\sigma_{\text{cr}} = N_{\text{cr}}/A = \pi^2 E/\lambda^2$. Stocky columns reach the yield stress before they buckle; slender columns buckle at stresses far below it. characterizes this effect.
2. **Buckling modes**: Higher modes correspond to buckling patterns with multiple half-waves. For $n = 2$, the beam develops two half-sine waves along its length, requiring four times the load of the first mode.
3. **Stability**: When $N < N_{\text{cr}}$, the beam is stable and returns to straight configuration if perturbed. When $N > N_{\text{cr}}$, the beam is unstable and buckles dramatically.

Effective length: the four Euler cases

For boundary conditions other than pin-pin, solving the buckling equation with the corresponding four boundary conditions (Chapter 6) gives the same formula with the length replaced by an **effective (buckling) length** $L_0 = KL$, the length of the equivalent half sine wave:

$$N_{\text{cr}} = \frac{\pi^2 E I_y}{(KL)^2} \tag{21}$$

where K is the effective length factor:

- **Pin-pin**: $K = 1.0$ (ends are free to rotate)
- **Fixed-free**: $K = 2.0$ (one end fixed, one free)
- **Fixed-fixed**: $K = 0.5$ (both ends fixed)
- **Fixed-pin**: $K = 0.7$ (one end fixed, one pinned)

Higher K values correspond to longer effective lengths and lower buckling loads. A clamped–clamped beam carries four times the load of a pinned–pinned one; a cantilever only a quarter. (The exact value for fixed–pin is $K \approx 0.699$.)

Example: a buckling MEMS bridge

A silicon bridge ($E = 169$ GPa) of length $L = 500$ μm, thickness $h = 2$ μm and width $b = 20$ μm is clamped at both ends. It buckles in the thin direction, with $I_y = bh^3/12 = 1.33 \times 10^{-23}$ m^4 and $A = bh = 4 \times 10^{-11}$ m^2. With $L_0 = L/2$,

$$N_{\text{cr}} = \frac{\pi^2 E I_y}{(L/2)^2} \approx 0.36 \text{ mN}, \qquad \varepsilon_{\text{cr}} = \frac{N_{\text{cr}}}{EA} = \frac{\pi^2 h^2}{3L^2} \approx 5 \times 10^{-5}.$$

A compressive strain of only 5×10^{-5} is enough to buckle the bridge. Residual stresses from fabrication or a thermal-expansion mismatch of a few tens of kelvin between the bridge and the substrate easily reach this value. Buckled bridges are a common failure in MEMS — and are also used on purpose, e.g. in bistable switches.

Buckling with initial imperfections

Real beams are never perfectly straight. With an initial deflection $w_0(x)$, the governing equation becomes:

$$E I_y w_{,xxxx} + N w_{,xx} = -N w_{0,xx} \tag{22}$$

This shows that initial imperfections are amplified by the axial load. The maximum deflection increases as N approaches the critical load, leading to failure through a gradual loss of stiffness rather than a sudden snap-through.

Chapter Summary

This chapter treated buckling as a loss of stability of an equilibrium:

- **Stability**: an equilibrium is stable, neutral or unstable; at the critical load the straight configuration becomes neutral
- **Second-order theory**: equilibrium in the deformed configuration, $M = Nw$ for a pin-pin column with compressive force N
- **Buckling equation**: $EI_y w'''' + Nw'' = q$; the term Nw'' softens the beam in compression
- **Euler load**: $N_{\text{cr}} = \pi^2 EI_y / L_0^2$ with the effective length $L_0 = KL$ ($K = 1$ pin-pin, 2 fixed-free, 0.5 fixed-fixed, ≈ 0.7 fixed-pin)
- **Buckling modes**: $N_n = n^2 N_{\text{cr}}$; the amplitude of the buckled shape is undetermined in linear theory
- **Critical stress**: $\sigma_{\text{cr}} = \pi^2 E / \lambda^2$ with the slenderness $\lambda = L_0 / r$, $r = \sqrt{I_y / A}$
- **Imperfections** are amplified as $N \to N_{\text{cr}}$; slender MEMS structures can buckle at strains of order 10^{-5}