# 15. Yield criteria

Learning Objectives

After completing this chapter, you should be able to:

- **Explain** what a yield (failure) criterion is and why it is formulated with invariants
- **Apply** the Tresca and von Mises criteria to ductile materials
- **Apply** the Coulomb–Mohr and Drucker–Prager criteria to pressure-dependent materials
- **Compare** the yield surfaces in principal stress space

Plastic failure occurs when a material transitions from elastic to plastic (inelastic) behavior. At the point of transition, the material begins to deform permanently under stress. To predict when this transition occurs, we use **yield criteria**, which are mathematical conditions defining the limit of elastic behavior in terms of the applied stress state.

## Introduction to Yield Criteria

A yield criterion is a function $F(\underline{\sigma})$ of the stress tensor that defines a surface in stress space. When the material is loaded elastically (inside or on the yield surface), $F(\underline{\sigma}) < 0$ or $F(\underline{\sigma}) = 0$. When the stress exceeds the yield surface, $F(\underline{\sigma}) > 0$, and the material begins to deform plastically.

The most important yield criteria are:

1. **Tresca** (maximum shear stress): ductile metals, simple to evaluate by hand
2. **Von Mises** (maximum distortion energy): the most widely used criterion for ductile metals
3. **Coulomb–Mohr**: materials whose strength depends on pressure (soils, rocks, concrete, brittle materials)
4. **Drucker–Prager**: a smooth version of Coulomb–Mohr, used for granular and geotechnical materials

Because the material is isotropic, a yield criterion cannot depend on the orientation of the coordinate system. It is therefore written in terms of **invariants** of the stress tensor, or equivalently of the principal stresses $\sigma_1 \geq \sigma_2 \geq \sigma_3$ (Chapter 14).

## Stress invariants (recap)

We recall the **stress invariants** from Chapter 14, which are scalar quantities derived from the stress tensor that remain unchanged under coordinate rotation. The three principal invariants of the stress tensor are:

$$I_1 = \mathrm{tr}(\underline{\sigma}) = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \sigma_1 + \sigma_2 + \sigma_3 \tag{1}$$

where $\sigma_1, \sigma_2, \sigma_3$ are the principal stresses.

$$I_2 = \frac{1}{2}\left[(\mathrm{tr}\underline{\sigma})^2 - \mathrm{tr}(\underline{\sigma}^2)\right] \tag{2}$$

$$I_3 = \det(\underline{\sigma}) \tag{3}$$

The **mean stress** (hydrostatic pressure) is:

$$\sigma_m = \frac{I_1}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \tag{4}$$

The **deviatoric stress tensor** represents the stress deviations from the mean stress:

$$\underline{s} = \underline{\sigma} - \sigma_m \underline{1} \tag{5}$$

where $\underline{1}$ is the identity matrix.

The **deviatoric stress invariant** (second invariant of deviatoric stress) is:

$$J_2 = \frac{1}{2}\mathrm{tr}(\underline{s}^2) = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] \tag{6}$$

The **equivalent stress** (or von Mises stress) is defined as:

$$\sigma_e = \sqrt{3J_2} \tag{7}$$

## Tresca yield criterion

Plastic deformation of metals happens by crystal planes sliding over each other, which is driven by shear stress. Henri Tresca (1814–1885) therefore proposed that a material yields when the **maximum shear stress** reaches a critical value. From Mohr's circles (Chapter 14), the maximum shear stress is $(\sigma_1 - \sigma_3)/2$. In a uniaxial tensile test ($\sigma_1 = \sigma$, $\sigma_2 = \sigma_3 = 0$) yield occurs at $\sigma = \sigma_y$, where $\tau_{\max} = \sigma_y/2$. The **Tresca criterion** is therefore

$$\boxed{\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_y}{2}} \qquad \text{or} \qquad \max\left(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|\right) = \sigma_y.$$

Like von Mises (below), Tresca does not depend on the hydrostatic stress: adding the same value to all three principal stresses does not change their differences. In principal stress space, the Tresca yield surface is a **hexagonal prism** around the hydrostatic axis $\sigma_1 = \sigma_2 = \sigma_3$. For plane stress ($\sigma_3 = 0$), it is a hexagon in the $\sigma_1$–$\sigma_2$ plane.

Tresca only uses the largest and the smallest principal stress, which makes it easy to evaluate by hand. Its drawback is the corners of the hexagon, where the direction of plastic flow is not uniquely defined.

## Von Mises yield criterion

Richard von Mises (1883–1953), and independently Huber and Hencky, proposed that yielding starts when the elastic energy stored in *changing the shape* of the material, the distortion energy, reaches a critical value. The distortion energy is proportional to $J_2$, the second invariant of the deviatoric stress.

The **von Mises yield criterion** is the most commonly used criterion for metals and states that plastic flow initiates when the deviatoric stress reaches a critical value:

$$\sigma_e = \sigma_y \quad \text{or} \quad \sqrt{3J_2} = \sigma_y \tag{8}$$

where $\sigma_y$ is the **yield stress** (uniaxial).

Equivalently, this can be written as:

$$\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = \sqrt{2}\sigma_y \tag{9}$$

In terms of the full stress tensor components:

$$\sigma_e = \sqrt{\frac{1}{2}\left[(\sigma_{xx}-\sigma_{yy})^2 + (\sigma_{yy}-\sigma_{zz})^2 + (\sigma_{zz}-\sigma_{xx})^2\right] + 3\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)} \tag{10}$$

**Key characteristics of the von Mises criterion:**

1. **Pressure independent**: The yield surface does not depend on the mean stress $\sigma_m$. This means that hydrostatic pressure alone (without shear) does not cause plastic flow.
2. **Cylindrical symmetry**: The yield surface in principal stress space is a cylinder aligned with the hydrostatic axis ($\sigma_1 = \sigma_2 = \sigma_3$).
3. **Isotropic**: The criterion applies equally in all directions, suitable for isotropic materials like metals.

**Uniaxial tension**: For a uniaxial stress state $\sigma_1 = \sigma$, $\sigma_2 = \sigma_3 = 0$:

$$\sigma_e = \sigma = \sigma_y$$

**Pure shear**: For pure shear $\sigma_1 = \tau$, $\sigma_2 = 0$, $\sigma_3 = -\tau$:

$$\sigma_e = \sqrt{3}\tau = \sigma_y \implies \tau = \frac{\sigma_y}{\sqrt{3}}$$

## Tresca and von Mises compared

Both criteria are pressure-independent and coincide for uniaxial tension, by construction. They differ most in **pure shear**, $\sigma_1 = \tau$, $\sigma_2 = 0$, $\sigma_3 = -\tau$:

| Criterion | Shear yield stress $\tau_y$ |
|---|---|
| Tresca | $\sigma_y/2 = 0.5\,\sigma_y$ |
| von Mises | $\sigma_y/\sqrt{3} \approx 0.577\,\sigma_y$ |

In principal stress space, the von Mises cylinder circumscribes the Tresca prism: Tresca is the more conservative criterion, and the difference is at most about 15%. Experiments on most ductile metals lie between the two and slightly closer to von Mises.

## Coulomb–Mohr criterion

For soils, rocks, concrete and many brittle materials, the strength depends strongly on pressure: they are much stronger in compression than in tension, and confinement makes them stronger. Charles-Augustin de Coulomb (1736–1806) described failure as sliding on a plane where the shear stress $\tau$ overcomes a **cohesion** $c$ plus an internal friction that grows with the normal compressive stress on the plane,

$$|\tau| = c - \sigma_n \tan\phi,$$

where $\sigma_n$ is the normal stress on the plane (negative in compression) and $\phi$ is the **angle of internal friction**. In the Mohr diagram this is a pair of straight lines (the failure envelope). A stress state is safe as long as its largest Mohr circle stays inside the envelope; failure occurs when the circle touches it. Expressed in principal stresses, with tension positive and $\sigma_1 \geq \sigma_3$,

$$\boxed{\sigma_1 \,(1 + \sin\phi) - \sigma_3 \,(1 - \sin\phi) = 2c\cos\phi.}$$

The uniaxial tensile and compressive strengths follow by setting $\sigma_3 = 0$ and $\sigma_1 = 0$, respectively,

$$\sigma_t = \frac{2c\cos\phi}{1 + \sin\phi}, \qquad \sigma_c = \frac{2c\cos\phi}{1 - \sin\phi},$$

so that the criterion can also be written as

$$\frac{\sigma_1}{\sigma_t} - \frac{\sigma_3}{\sigma_c} = 1.$$

For $\phi > 0$ the compressive strength is larger than the tensile strength. For $\phi = 0$ the envelope is horizontal, $|\tau| = c$, and Coulomb–Mohr reduces to Tresca with $c = \sigma_y/2$. In principal stress space, the Coulomb–Mohr surface is an irregular hexagonal **pyramid** that opens towards compression.

## Drucker–Prager criterion

The **Drucker-Prager yield criterion** generalizes the von Mises criterion to account for the effect of mean stress, making it particularly useful for materials like soil, concrete, and rock where the yield stress depends on hydrostatic pressure.

The Drucker-Prager yield criterion is defined as:

$$F(\underline{\sigma}) = \sqrt{J_2} + \alpha I_1 - k = 0 \tag{11}$$

or equivalently:

$$\sqrt{J_2} + \alpha\sigma_m - k = 0 \tag{12}$$

where $\alpha$ and $k$ are material parameters that must be determined from experiments.

**Physical interpretation:**

- $\sqrt{J_2}$ represents the magnitude of deviatoric stress (similar to von Mises)
- $\alpha I_1$ (or $\alpha\sigma_m$) represents the pressure-dependent component
- When $\alpha = 0$, the criterion reduces to von Mises
- When $\alpha > 0$, the yield stress increases with hydrostatic pressure (typical for granular materials)

**Determining material parameters:**

The parameters $\alpha$ and $k$ are typically determined from two simple tests:

1. **Uniaxial compression** with strength $\sigma_c > 0$: the only non-zero principal stress is $-\sigma_c$

   At yield: $I_1 = -\sigma_c$, $J_2 = \frac{\sigma_c^2}{3}$

   $$\frac{\sigma_c}{\sqrt{3}} - \alpha\sigma_c - k = 0$$

2. **Uniaxial tension** with strength $\sigma_t$: the only non-zero principal stress is $\sigma_t$

   At yield: $I_1 = \sigma_t$, $J_2 = \frac{\sigma_t^2}{3}$

   $$\frac{\sigma_t}{\sqrt{3}} + \alpha\sigma_t - k = 0$$

From these two equations, we can solve for $\alpha$ and $k$:

$$\alpha = \frac{\sigma_c - \sigma_t}{\sqrt{3}\,(\sigma_c + \sigma_t)}, \qquad k = \frac{2\,\sigma_c\,\sigma_t}{\sqrt{3}\,(\sigma_c + \sigma_t)}. \tag{13}$$

For $\sigma_c = \sigma_t = \sigma_y$ we recover von Mises: $\alpha = 0$ and $k = \sigma_y/\sqrt{3}$.

**Special case: Cohesive-frictional material**

Drucker–Prager is a smooth approximation of Coulomb–Mohr. If cohesion $c$ and friction angle $\phi$ are known, a common choice (matching Coulomb–Mohr under plane strain) is:

$$\alpha = \frac{\tan\phi}{\sqrt{9 + 12\tan^2\phi}} \tag{14}$$

$$k = \frac{3c}{\sqrt{9 + 12\tan^2\phi}} \tag{15}$$

**Comparison with von Mises:**

The Drucker-Prager criterion is more general than von Mises:

- **Von Mises**: Pressure-independent, $\sigma_y$ constant regardless of hydrostatic stress
- **Drucker-Prager**: Pressure-dependent, yield stress increases with confining pressure (positive $\alpha$)

This makes Drucker-Prager more suitable for: - Soils and rocks (where cohesion and friction are important) - Concrete and other brittle materials - Powder-like materials - Any material where compressive strength exceeds tensile strength

## Yield Surface Geometry

In principal stress space:

| Criterion | Pressure dependent? | Shape in principal stress space |
|---|---|---|
| Tresca | no | regular hexagonal prism around the hydrostatic axis |
| von Mises | no | circular cylinder around the hydrostatic axis |
| Coulomb–Mohr | yes | irregular hexagonal pyramid |
| Drucker–Prager | yes | circular cone |

The prism and the cylinder are the pressure-independent limits ($\phi = 0$, $\alpha = 0$) of the pyramid and the cone.

The Drucker-Prager cone opens wider with increasing pressure, indicating that higher hydrostatic pressure increases the allowable shear stress before yielding.

## Which criterion for which material?

Experiments and simulations show that:

- **Ductile materials** (most metals) start to deform plastically when the shear stress is large. Tresca and von Mises describe the **onset of yield** well; von Mises is the standard choice in engineering codes.
- **Pressure-sensitive materials** (soils, rocks, concrete, ceramic powders, many polymers) are described by Coulomb–Mohr or Drucker–Prager.
- **Brittle materials** (glass, silicon, cast iron) do not yield but **fracture**, typically when the largest tensile principal stress reaches the tensile strength, $\sigma_1 = \sigma_t$ (the *maximum principal stress* or Rankine criterion). Their failure is controlled by cracks and is the subject of fracture mechanics.

## Practical applications

**Engineering design**: The yield criterion determines the maximum allowable stress before plastic deformation initiates. Safety factors are typically applied:

$$\sigma_e \leq \frac{\sigma_y}{n_s} \tag{16}$$

where $n_s$ is the safety factor (typically 1.5-3.0 depending on application).

**Material selection**: Materials with high yield stress are preferred for structural applications to limit plastic deformation and ensure elastic behavior under service loads.

Chapter Summary

This chapter introduced criteria that predict when a material stops behaving elastically:

- **Yield criterion**: a function $F(\underline{\sigma})$ of the stress invariants; $F < 0$ elastic, $F = 0$ yield
- **Tresca**: $\tau_{\max} = (\sigma_1 - \sigma_3)/2 = \sigma_y/2$; hexagonal prism; pressure-independent
- **Von Mises**: $\sigma_e = \sqrt{3J_2} = \sigma_y$; circular cylinder; pressure-independent; pure shear yield at $\tau_y = \sigma_y/\sqrt{3}$
- **Coulomb–Mohr**: $|\tau| = c - \sigma_n \tan\phi$, or $\sigma_1(1+\sin\phi) - \sigma_3(1-\sin\phi) = 2c\cos\phi$; stronger in compression than in tension
- **Drucker–Prager**: $\sqrt{J_2} + \alpha I_1 = k$; smooth cone, reduces to von Mises for $\alpha = 0$
- **Choice of criterion**: Tresca/von Mises for ductile metals, Coulomb–Mohr/Drucker–Prager for pressure-sensitive materials, maximum principal stress for brittle fracture
- **Safety factor**: design requires $\sigma_e \leq \sigma_y/n_s$