14. Stress transformation and Mohr's circle

Learning Objectives

After completing this chapter, you should be able to:

- **Rotate** the stress tensor, $\underline{\sigma}' = \underline{R}^T \underline{\sigma}\, \underline{R}$
- **Calculate** principal stresses and principal directions as eigenvalues and eigenvectors of $\underline{\sigma}$
- **Compute** the stress invariants and the hydrostatic and deviatoric parts of the stress
- **Construct** Mohr's circle and read off the maximum shear stress

Vectors and tensors

Note

We have encountered vectors in the first chapters on statics of rigid bodies and have intuitively worked with them. To recap, a vector is an object that represents a direction and a magnitude. Geometrically, they are often represented as arrows. In a Cartesian coordinate system (or basis), a vector can be represented by a set of numbers. For example in two dimensions, we denote the components of the vector $\vec{v}$ as the x and y components and typically write the vector in the column-form

$$\vec{v} = v_x \hat{x} + v_y \hat{y} \equiv \begin{pmatrix} v_x \\ v_y \end{pmatrix} \tag{1}$$

where v_x and v_y are real numbers that are called the * components * of the vector $\vec{v}$. $\hat{x}$ and $\hat{y}$ are vectors of unit length that point along the x- and y-directions of the coordinates. The length of the vector is given by the 2-norm $|\vec{v}| \equiv \|\vec{v}\|_2 = (v_x^2 + v_y^2)^{1/2}$. It is important to emphasize that this is a * representation * of the vector and the components $v_i \in \mathbb{R}$ depend on the specific basis $\hat{x}$ and $\hat{y}$. This representation is called a * tensor of order 1. *The order of a tensor is sometimes also called degree or rank. Any component-wise*

representation, such as the one on the right hand side of Eq. (1), implies a basis. The basis is written explicitly as $\hat{x}$ and $\hat{y}$ in the middle expression of Eq. (1). Note that in component notation, the basis vectors are

$$\hat{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \textit{and} \quad \hat{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2}$$

We will restrict the discussion here to orthogonal bases where $\hat{x} \cdot \hat{y} = 0$. The discussion here will use two-dimensional examples, but a generalization to more that two dimensions is straightforward Formally, a vector is an element of a vector space. *A vector space V (often also called a* linear space*) is a set of objects (for example the set containing our basis vectors $\hat{x}$ and $\hat{y}$ and linear combinations thereof) along with two operations: Addition (of two vectors) and multiplication (of a vector) with a scalar. These operations again yield a vector, i.e. an element of V. This can be expressed as

- $\vec{u},\ \vec{v} \in V$, then $\vec{u} + \vec{v} \in V$
- $a \in \mathbb{R}$ and $\vec{u} \in V$, then $a \cdot \vec{u} \in V$ and hence Eq. (1) yields a vector. In general, we may multiply the vectors by an element of a * algebraic number field *$\mathbb{F}$ rather than $\mathbb{R}$. Then we say V is a vector space over the field $\mathbb{F}$. In these notes (and our lectures) we will always deal with either real ($\mathbb{F} = \mathbb{R}$) or complex ($\mathbb{F} = \mathbb{C}$) numbers.

Recall the concept of an algebraic number field in mathematics. A field is an algebraic structure that is a set along with two operations "+" and "$ $" associating an element with two elements of the set. The operations are required to satisfy the field axioms: - Associativity of addition and multiplication: $a+(b+c) = (a+b)+c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

- Commutativity of addition and multiplication: $a + b = b + a$ and $a \cdot b = b \cdot a$.
- Additive and multiplicative identity: $0 \in \mathbb{F}$ with $a + 0 = a$ and $1 \in \mathbb{F}$ with $1 \cdot a = a$.
- Additive inverses: $\forall a \in \mathbb{F}$ we have an inverse element $i = -a$ with $a+i = a+(-a) = 0$
- Multiplicative inverses: $\forall a \neq 0$ we have an element a^{-1} with $a \cdot a^{-1} = 1$.
- Distributivity of multiplication over addition: $a \cdot (b + c) = a \cdot b + a \cdot c$ Note that in physics a * field * is typically a quantity that depends on position and not an algebraic number field. A scalar field $\phi(\vec{r})$ would be scalar quantity $\phi \in \mathbb{R}$ that depends on a vector (the position) $\vec{r} \in V$. While these two meanings of the term * field * exist, we will always refer to this latter physical meaning in the following. Additionally, we have here used the dot $ $ to indicate a scalar multiplication, but we will reserve this dot in the rest of these notes for contractions, i.e. operations that implicitly include a sum.

The simplest (and also most important) operation on a vector is a linear transformation. The simplest linear transformation is the multiplication with a scalar $a \in \mathbb{R}$. In terms of the geometric interpretation of a vector, this multiplication would change the length of the vector by a but not its direction.
A general linear transformation can in addition change the direction of the vector and therefore represent for example rotations. Given a vector $\vec{u}$ and a scalar \$ \$, a linear transformation $\mathcal{L}$ to a vector $\vec{u}' = \mathcal{L}\vec{u}$ has the properties:

$$\mathcal{L}(\alpha\vec{u}) = \alpha\mathcal{L}\vec{u} \tag{3}$$
$$\mathcal{L}(\vec{u} + \vec{v}) = \mathcal{L}\vec{u} + \mathcal{L}\vec{v} \tag{4}$$

We will only deal with linear operations that map between the same vector space V, $\mathcal{L} : V \mapsto V$. (It may map between quantities with different physical units.) Given a component-wise (Cartesian) representation of a vector, Eq. (1), a linear transformation can be expressed as a multiplication by a matrix. This is easily by applying Eqs. (3) and (4) to

$$\vec{w} = \mathcal{L}\vec{v} = \mathcal{L}(v_x\hat{x} + v_y\hat{y}) = v_x\mathcal{L}\hat{x} + v_y\mathcal{L}\hat{y}. \tag{5}$$

We can express any element of our vector space as a linear combination of the basis vectors, hence also

$$\mathcal{L}\hat{x} = L_{xx}\hat{x} + L_{yx}\hat{y} \tag{6}$$
$$\mathcal{L}\hat{y} = L_{xy}\hat{x} + L_{yy}\hat{y}. \tag{7}$$

Application of the linear operation to an arbitrary vector, Eq. (5), can therefore be expressed as

$$\vec{w} = \mathcal{L}\vec{v} = (L_{xx}v_x + L_{xy}v_y)\hat{x} + (L_{yx}v_x + L_{yy}v_y)\hat{y} \equiv \underline{L} \cdot \vec{v} \tag{8}$$

where $\underline{L} \cdot \vec{v}$ is the multiplication of the matrix $\underline{L}$ with the vector $\vec{v}$. A matrix is therefore a * representation * of a linear operation. Note that from Eq. (6) it is straightforward to see that formally we obtain the components of the matrix from

$$L_{ij} = \hat{\imath} \cdot \mathcal{L}\hat{\jmath}. \tag{9}$$

Here the \$ \$ indicates the inner product or contraction of two vectors. In component-wise notation we can express Eq. (8) as

$$w_i = \sum_{j=x,y} L_{ij}v_j \equiv L_{ij}v_j \tag{10}$$

where the last term on the right-hand side uses the * Einstein summation * convention. In this convention, a summation over repeated indices within the same quantity or in

products is implicit. This summation is also often called a * contraction * and in dyadic notation it is indicated by a centered dot, e.g. $\vec{w} = \underline{L} \cdot \vec{v}$. It is straighforward to "convert" from dyadic notation to component-wise or * index notation. *Imagine the i, j component of the resultant matrix of the product*

$$[\underline{A} \cdot \underline{B} \cdot \underline{C}]_{ij} = A_{ik} B_{kl} C_{lj}. \tag{11}$$

Converting from the dyadic notation to index notation involves identifying the indices of the resultant (first index of the first matrix in the product i and last index of the last matrix in the product j) and introducing repeated indices for the summation. These indices, k and l in the example, always sit next to each other and are the indices that are contracted*. The advantage of the index notation is that it is unambiguous, but it may hide the physical structure of the underlying operations. In these notes, we will therefore intermix "dyadic" notation as in Eq. (8) and index notation as appropriate. Note that there is also a double contraction, for example

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl}, \tag{12}$$

that in dyadic notation we would indicate with two dots: $\underline{\sigma} = \underline{\underline{C}} : \underline{\varepsilon}$. Each dot therefore represents a sum over some (hidden) index.

Rotating the stress tensor

A tensor is a representation of a linear transformation: A tensor of order 2 (a matrix) transforms a tensor of order 1 (a vector) into another tensor of order 1. We have already encountered the Cauchy stress tensor $\underline{\sigma}$ of solid mechanics. It transforms an area vector $\vec{A}$ into a force vector $\vec{F}$,

$$\vec{F} = \underline{\sigma} \cdot \vec{A} \quad \text{or in index notation} \quad F_i = \sigma_{ij} A_j. \tag{13}$$

We describe $\vec{F}$, $\vec{A}$ and $\underline{\sigma}$ in terms of their components, i.e. we describe a realization of the linear transformation (and the area and force vectors).

A rotation does not affect the effect of the linear transformation. Hence, if we change the coordinate system from $\hat{x}$, $\hat{y}$ to $\hat{x}'$, $\hat{y}'$, then the component of our vector $\vec{F}$ change to

$$F'_x = \vec{F} \cdot \hat{x}' = F_x \hat{x} \cdot \hat{x}' + F_y \hat{y} \cdot \hat{x}' \tag{14}$$

$$F'_y = \vec{F} \cdot \hat{y}' = F_y \hat{x} \cdot \hat{y}' + F_y \hat{y} \cdot \hat{y}'. \tag{15}$$

Comparing with Eq. (**??**) yields the compact notation

$$\vec{F}' = \underline{R}^T \cdot \vec{F} \quad \text{or in index notation} \quad F'_i = F_j R_{ji}. \tag{16}$$

Note that the transpose shows up because $\underline{R}$ describes the rotation of the basis and we are here rotating a vector expressed within this basis, which is the inverse operation.

Since we now understand how to rotate vectors, we can ask the question of how to rotate a tensor of order 2. Starting from Eq. (13) and using the inverse of Eq. (16), we write

$$\underline{R} \cdot \vec{F}' = \underline{\sigma} \cdot \left(\underline{R} \cdot \vec{A}' \right), \tag{17}$$

and multiply by $\underline{R}^T$ from the left to yield

$$\vec{F}' = \left(\underline{R}^T \cdot \underline{\sigma} \cdot \underline{R} \right) \cdot \vec{A}'. \tag{18}$$

In the rotated coordinate system, the tensor attains the representation

$$\underline{\sigma}' = \underline{R}^T \cdot \underline{\sigma} \cdot \underline{R} \quad \text{or in index notation} \quad \sigma'_{ij} = \sigma_{kl} R_{ki} R_{lj} \tag{19}$$

since this leave the expression for the linear transformation

$$\vec{F}' = \underline{\sigma}' \cdot \vec{A}'. \tag{20}$$

invariant. The trace and determinant of the rotated tensor are

$$\mathrm{tr}\underline{\sigma}' = \mathrm{tr}\left(\underline{R}^T \cdot \underline{\sigma} \cdot \underline{R} \right) = \mathrm{tr}\left(\underline{R} \cdot \underline{R}^T \cdot \underline{\sigma} \right) = \mathrm{tr}\underline{\sigma} \tag{21}$$

$$\det \underline{\sigma}' = \det\left(\underline{R}^T \cdot \underline{\sigma} \cdot \underline{R} \right) = \det\left(\underline{R} \cdot \underline{R}^T \cdot \underline{\sigma} \right) = \det \underline{\sigma} \tag{22}$$

and hence invariant under rotation. Note that in general for an $n \times n$ tensor, there are n invariants; more on this will be discussed below when talking about eigenvalues.

Since we now understand how to rotate tensors of order 2, we can ask the question how to rotate a tensor of order 4. As an example, we use the stiffness tensor $\underline{\underline{C}}$, that we encountered in Chapter 13. This tensor transforms a strain tensor $\underline{\varepsilon}$ into a stress tensor $\underline{\sigma}$,

$$\underline{\sigma} = \underline{\underline{C}} : \underline{\varepsilon} \quad \text{or in index notation} \quad \sigma_{ij} = C_{ijkl} \varepsilon_{kl}. \tag{23}$$

We can rewrite this using the inverse of Eq. (19) as

$$\underline{R} \cdot \underline{\sigma}' \cdot \underline{R}^T = \underline{\underline{C}} : \left(\underline{R} \cdot \underline{\varepsilon}' \cdot \underline{R}^T \right) \quad \text{or} \quad R_{ik} R_{jl} \sigma'_{kl} = C_{ijkl} R_{km} R_{ln} \varepsilon'_{mn}, \tag{24}$$

which we now multiply from the left with $\underline{R}^T$ and from the right with $\underline{R}$. This gives

$$\sigma'_{ij} = C_{mnop} R_{mi} R_{nj} R_{ok} R_{pl} \varepsilon'_{kl}, \tag{25}$$

and hence the transformation rule

$$C'_{ijkl} = C_{mnop} R_{mi} R_{nj} R_{ok} R_{pl}. \tag{26}$$

Quantities that transform as Eqs. (16), (19) and (26) are called * tensors*.

Principal stresses

Let us discuss in more detail what happens if we rotate a symmetric tensor of order 2, i.e. a tensor that fulfills $\underline{\sigma}^T = \underline{\sigma}$. From Eq. (19) it is straightforward to see, that $\underline{\sigma}'^T = \underline{\sigma}'$, i.e. the transformed tensor is also symmetric.

We now explicitly write the rotation for the tensor

$$\underline{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix}, \tag{27}$$

using the rotation

matrix Eq. (**??**). This gives symmetric $\underline{\sigma}' = \underline{R}^T \cdot \underline{\sigma} \cdot \underline{R}$ with the components

$$\sigma'_{xx} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \frac{1}{2}(\sigma_{xx} - \sigma_{yy})\cos 2\theta + \sigma_{xy}\sin 2\theta \tag{28}$$

$$\sigma'_{yy} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) - \frac{1}{2}(\sigma_{xx} - \sigma_{yy})\cos 2\theta - \sigma_{xy}\sin 2\theta \tag{29}$$

$$\sigma'_{xy} = -\frac{1}{2}(\sigma_{xx} - \sigma_{yy})\sin 2\theta + \sigma_{xy}\cos 2\theta. \tag{30}$$

There is a specific rotation angle θ_0 where the diagonal elements σ'_{xx} and σ'_{yy} become extremal. It is determined from $\sigma'_{xx,\theta} = \sigma'_{yy,\theta} = 0$,

$$\tan 2\theta_0 = \frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}. \tag{31}$$

At this rotation angle we find that the off-diagonal components vanish, $\sigma'_{xy}(\theta_0) = 0$, and the rotated matrix is diagonal,

$$\underline{\sigma} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \tag{32}$$

with diagonal elements

$$\sigma_1 = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \left[\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2\right]^{1/2} \tag{33}$$

$$\sigma_2 = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \left[\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2\right]^{1/2}. \tag{34}$$

This is the simplest example of the diagonalization of a matrix. The elements of the diagonalized Cauchy stress tensor, σ_1 and σ_2, are called the * principal stresses*.

The diagonalization of a symmetric matrix always corresponds to the rotation into a new coordinate system. We have explictly shown this for the two-dimensional case here and will now show it in more generality for the three-dimensional case. In this process, we will encounter the concept of a stress invariant.

Stress invariants

Equation (32) fulfills the eigenvalue equations $\underline{\sigma} \cdot \hat{x} = \sigma_1 \hat{x}$ and $\underline{\sigma} \cdot \hat{y} = \sigma_2 \hat{y}$. Rather than explicitly computing a rotation, we can ask the question whether we can find a scalar \$ \$ and a vector $\vec{v}$ that fulfills the eigenvalue equation

$$\underline{\sigma} \cdot \vec{v} = \lambda \vec{v}. \tag{35}$$

This equation of course has the trivial solution $\vec{v} = 0$. It can only have a nontrivial solution if

$$\det\left(\underline{\sigma} - \lambda \underline{1}\right) = 0. \tag{36}$$

For a $n \times n$ matrix, Eq. (36) leads to a polynomial of order n in \$ \$ with n (possibly complex valued) solutions.

For the case of a symmetric 3×3 matrix, we can write this down explicitly as

$$\det \begin{pmatrix} \sigma_{xx} - \lambda & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \lambda & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \lambda \end{pmatrix} = -\lambda^3 + I_1 \lambda^2 - I_2 \lambda + I_3 = 0 \tag{37}$$

with

$$I_1 = \mathrm{tr}\underline{\sigma} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \tag{38}$$

$$I_2 = \sigma_{yy}\sigma_{zz} + \sigma_{xx}\sigma_{zz} + \sigma_{xx}\sigma_{yy} - \sigma_{yz}^2 - \sigma_{xz}^2 - \sigma_{xy}^2 \tag{39}$$

$$I_3 = \det \underline{\sigma} \tag{40}$$

The quantities I_1 to I_3 are called * invariants*. We have shown above explicitly that the trace and the determinant are invariant under rotation. The same holds true for all coefficients of the characteristic polynomial (two of which are actually trace and determinant). This is because

$$\det\left(\underline{R}^T \cdot \underline{\sigma} \cdot \underline{R} - \lambda \underline{1}\right) = \det\left[\underline{R}^T \cdot \left(\underline{\sigma} - \lambda \underline{1}\right) \cdot \underline{R}\right] = \det\left(\underline{\sigma} - \lambda \underline{1}\right). \tag{41}$$

The 3-dimensional tensor therefore has three invariants. These invariants have important physical interpretations. For the stress tensor, I_1 is related to the hydrostatic stress and I_2 to the shear stress.

Note that for a diagonal matrix,

$$\underline{\sigma}' = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}, \tag{42}$$

the invariants are

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3 \tag{43}$$
$$I_2 = \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3 \tag{44}$$
$$I_3 = \sigma_1\sigma_2\sigma_3. \tag{45}$$

By equating Eqs. (38) to (40) with Eqs. (43) to (45) we can calculate the eigenvalues σ_1, σ_2 and σ_3.

Once we have computed the eigenvalues, we can obtain the corresponding eigenvectors by solving

$$\underline{\sigma}\vec{v}_1 = \sigma_1\vec{v}_1, \quad \underline{\sigma}\vec{v}_2 = \sigma_2\vec{v}_2, \quad \text{and} \quad \underline{\sigma}\vec{v}_3 = \sigma_3\vec{v}_3. \tag{46}$$

Note that the expressions only determine the direction of

$\vec{v}_i$, not its length, and we are free to require $|\vec{v}_i| = 1$. Furthermore, let us regard scalar products $\vec{v}_1 \cdot \vec{v}_2$, then

$$\sigma_1\vec{v}_1 \cdot \vec{v}_2 = (\underline{\sigma} \cdot \vec{v}_1) \cdot \vec{v}_2 = \vec{v}_1 \cdot (\underline{\sigma}^T \cdot \vec{v}_2) = \vec{v}_1 \cdot (\underline{\sigma} \cdot \vec{v}_2) = \sigma_2\vec{v}_1 \cdot \vec{v}_2 \tag{47}$$

and if $\sigma_1 \neq \sigma_2$ we must have $\vec{v}_1 \cdot \vec{v}_2 = 0$. Hence the eigenvectors of a symmetric matrix are orthonormal, or in other words, they form the basis of a coordinate system.

We can write Eq. (46) in the more compact notation

$$\underline{\sigma} \cdot \underline{R} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \cdot \underline{R} \tag{48}$$

with $\underline{R} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$. Multiplying from the left with $\underline{R}^{-1} = \underline{R}^T$ (this holds because the eigenvectors are orthonormal) we get

$$\underline{R}^T \cdot \underline{\sigma} \cdot \underline{R} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}. \tag{49}$$

This is nothing else than a coordinate transformation (rotation) of the tensor $\underline{\sigma}$. Since the diagonalization of a symmetric matrix leads to orthonormal eigenvectors, the diagonalization is a rotation of the coordinate system.

Hydrostatic and deviatoric stress

We call $\sigma_h = \mathrm{tr}\underline{\sigma}/3 = I_1/3$ the * hydrostatic stress*. It is the stress measure that tells us about volume expansion and contraction. The pressure is the negative of the hydrostatic stress,

$p = -\sigma_h$. Our sign convention is such that positive stresses (negative pressures) mean tension and negative stresses (positive pressures) compression.

Using the hydrostatic stress, we can construct yet another stress tensor that quantifies the * deviation * from a pure hydrostatic condition with stress $\sigma_h \underline{1}$. We define

$$\underline{s} = \underline{\sigma} - \sigma_h \underline{1}, \tag{50}$$

the * deviatoric stress*. Note that this tensor is constructed such that $\mathrm{tr}\underline{s} = 0$.

The invariants of the deviatoric stress are commonly denoted by the symbol J. We already know that $J_1 = 0$ by construction. The second invariant is given by

$$\begin{aligned} J_2 &= \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2\right] \\ &= \frac{1}{6}\left[(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{xx} - \sigma_{zz})^2 + 6(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{xz}^2)\right]. \end{aligned} \tag{51}$$

From this invariant we can derive the * von-Mises stress ${}^*\sigma_{vM} = \sqrt{3J_2}$. The von-Mises stress characterizes the pure shear contribution to the stress. The second invariant of the deviatoric stress plays an important role in plasticity models, where it is often assumed that a material flows when the von-Moses stress exceeds a certain threshold.

Example: Principal stresses from stress tensor

Tip

Consider a 2D stress state in the xy-plane:

$$\underline{\sigma} = \begin{pmatrix} 80 & 30 & 0 \\ 30 & 20 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ MPa}$$

Find:

- The principal stresses $\sigma_1, \sigma_2, \sigma_3$
- The maximum shear stress $\tau_{\max}$

Solution:
The given stress tensor describes a 2D stress state: there are normal stresses $\sigma_{xx} = 80$ MPa and $\sigma_{yy} = 20$ MPa, plus a shear stress $\tau_{xy} = 30$ MPa. The zeros in the third row and column indicate no stress components involving the z-direction (plane stress condition). Our goal is to find the *principal stresses*—the normal stresses that would act on specially oriented planes where the shear stress vanishes. These are the eigenvalues of the stress tensor.

Principal stresses are found by solving the eigenvalue equation $\det(\underline{\sigma} - \sigma \underline{I}) = 0$. Since the third row and column are all zeros except for the diagonal, the z-direction is already a principal direction with $\sigma_3 = 0$. We only need to find the eigenvalues of the 2D submatrix:

$$\underline{\sigma}_{2D} = \begin{pmatrix} 80 & 30 \\ 30 & 20 \end{pmatrix}$$

The eigenvalue equation becomes:

$$\det \begin{pmatrix} 80 - \sigma & 30 \\ 30 & 20 - \sigma \end{pmatrix} = 0$$

For a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant is $ad - bc$. Applying this:

$$(80 - \sigma)(20 - \sigma) - (30)(30) = 0$$

Expanding the first term:

$$(80 - \sigma)(20 - \sigma) = 80 \cdot 20 - 80\sigma - 20\sigma + \sigma^2 = 1600 - 100\sigma + \sigma^2$$

Subtracting $30^2 = 900$:

$$\sigma^2 - 100\sigma + 1600 - 900 = 0$$

$$\sigma^2 - 100\sigma + 700 = 0$$

This is the *characteristic equation* of the stress tensor. We solve it using the quadratic formula: for $a\sigma^2 + b\sigma + c = 0$, the solutions are $\sigma = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Here $a = 1$, $b = -100$, $c = 700$:

$$\sigma = \frac{100 \pm \sqrt{(-100)^2 - 4(1)(700)}}{2(1)} = \frac{100 \pm \sqrt{10000 - 2800}}{2} = \frac{100 \pm \sqrt{7200}}{2}$$

Computing $\sqrt{7200} = \sqrt{3600 \cdot 2} = 60\sqrt{2} \approx 84.85$:

$$\sigma = \frac{100 \pm 84.85}{2}$$

This gives two solutions:

$$\sigma_1 = \frac{100 + 84.85}{2} = \frac{184.85}{2} = 92.43 \text{ MPa}$$

$$\sigma_2 = \frac{100 - 84.85}{2} = \frac{15.15}{2} = 7.57 \text{ MPa}$$

By convention, we label principal stresses such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$. So we have $\sigma_1 = 92.43$ MPa (largest, tensile), $\sigma_2 = 7.57$ MPa (intermediate, tensile), and $\sigma_3 = 0$ MPa (smallest, from the z-direction).

The maximum shear stress occurs on planes oriented at 45° to the principal directions. Its magnitude is:

$$\tau_{\max} = \frac{\sigma_{\max} - \sigma_{\min}}{2} = \frac{\sigma_1 - \sigma_3}{2} = \frac{92.43 - 0}{2} = 46.21 \text{ MPa}$$

Note that we use σ_1 and σ_3 (not σ_2) because the maximum shear stress depends on the difference between the *largest* and *smallest* principal stresses.
The stress tensor has been transformed to principal coordinates where the largest tensile stress is 92.43 MPa (acting in a direction rotated from the original x-axis), a moderate tensile stress of 7.57 MPa acts perpendicular to it, no stress acts perpendicular to the xy-plane (this is the plane stress condition), and the maximum shear stress is 46.21 MPa (acting at $\pm 45°$ to the principal directions).
We can verify our answer using the invariants. The trace (sum of eigenvalues) should equal $\sigma_1+\sigma_2+\sigma_3 = 92.43+7.57+0 = 100$ MPa, which matches $\sigma_{xx}+\sigma_{yy}+\sigma_{zz} = 80+20+0 = 100$ MPa.

Mohr's circle

The rotation formulas for the 2D stress tensor have a simple geometric interpretation. Writing $\sigma_h = (\sigma_{xx} + \sigma_{yy})/2$ for the mean normal stress and

$$R = \left[\left(\frac{\sigma_{xx} - \sigma_{yy}}{2} \right)^2 + \sigma_{xy}^2 \right]^{1/2},$$

the components in the rotated system satisfy

$$(\sigma'_{xx} - \sigma_h)^2 + (\sigma'_{xy})^2 = R^2$$

for every rotation angle θ (square and add the formulas for σ'_{xx} and σ'_{xy}). In a diagram with the normal stress σ on the horizontal axis and the shear stress τ on the vertical axis, all stress states $(\sigma'_{xx}, \sigma'_{xy})$ that we obtain by rotating the coordinate system lie on a circle: **Mohr's circle**, with center σ_h and radius R. Rotating the coordinate system by θ moves the point along the circle by the angle 2θ.

Construction.

1. Mark the point $X = (\sigma_{xx}, \sigma_{xy})$, which represents the stresses on the face with normal x, and the point $Y = (\sigma_{yy}, -\sigma_{xy})$ for the face with normal y.
2. The segment XY is a diameter: its midpoint on the σ-axis is the center σ_h, its half-length the radius R.
3. The intersections with the σ-axis are the principal stresses $\sigma_{1,2} = \sigma_h \pm R$, where the shear stress vanishes. The angle between X and σ_1 on the circle is $2\theta_0$.

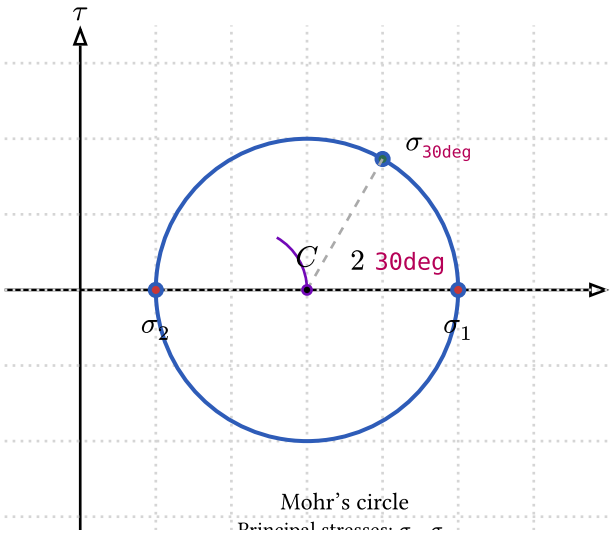


Figure 1: Mohr's circle provides a geometric representation of stress states in 2D. The horizontal axis represents normal stress σ, and the vertical axis represents shear stress τ. Principal stresses σ_1 (major) and σ_2 (minor) lie on the horizontal axis. The center of the circle is the hydrostatic stress $\sigma_h = (\sigma_1 + \sigma_2)/2$, and the radius is $R = (\sigma_1 - \sigma_2)/2$. Any point on the circle represents a stress state $(\sigma_\theta, \tau_\theta)$ at some angle θ, where the angle on the circle is 2θ (twice the physical angle). Maximum shear stress $\tau_{\max} = R$ occurs at $\theta = 45°$ (top and bottom of the circle).

4. The top and bottom of the circle give the **maximum shear stress** $\tau_{\max} = R = (\sigma_1 - \sigma_2)/2$, on planes rotated by 45° with respect to the principal directions.

Uniaxial tension. A bar pulled along x has $\sigma_{xx} = \sigma_0$ and all other components zero. Mohr's circle passes through 0 and σ_0; its radius is $\sigma_0/2$. Although only a normal stress is applied, planes inclined at 45° carry the shear stress $\tau_{\max} = \sigma_0/2$. This is why ductile metals in a tensile test often slip along planes at about 45° to the load: plastic deformation is driven by shear (Chapter 15).

Three dimensions. With three principal stresses $\sigma_1 \geq \sigma_2 \geq \sigma_3$ there are three circles, spanned by each pair of principal stresses. All possible stress states on any plane lie in the region between the largest circle and the two smaller ones, and the maximum shear stress is

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2}.$$

i Chapter Summary

This chapter described how the stress tensor changes when the coordinate system is rotated:

- **Tensors** are defined by their transformation rule; a second-order tensor rotates as $\underline{\sigma}' = \underline{R}^T \underline{\sigma}\, \underline{R}$
- **Principal stresses** are the eigenvalues of $\underline{\sigma}$; in the principal coordinate system the shear stresses vanish
- **Stress invariants** ($I_1 = \operatorname{tr} \underline{\sigma}$, I_2, $I_3 = \det \underline{\sigma}$) do not depend on the coordinate system
- **Hydrostatic and deviatoric stress**: $\underline{\sigma} = \sigma_h \underline{1} + \underline{s}$ with $\sigma_h = I_1/3$
- **Mohr's circle**: center σ_h, radius R; principal stresses $\sigma_h \pm R$ and maximum shear stress $\tau_{\max} = R = (\sigma_1 - \sigma_3)/2$
- **Uniaxial tension** produces the maximum shear stress $\sigma_0/2$ on planes at 45°