13. Hooke's law and plane problems

Learning Objectives

After completing this chapter, you should be able to:

- **Explain** the microscopic origin of elasticity in crystals, polymers and rubbers
- **Decompose** stress and strain into volumetric and deviatoric parts and relate them through K and G
- **Convert** between the elastic constants E, ν, K, G and λ
- **Write** the general and the isotropic Hooke's law $\underline{\sigma} = \underline{\underline{C}} : \underline{\varepsilon}$
- **Apply** plane stress and plane strain conditions

The equations we have derived so far are of two different kinds. **Balance (conservation) equations** — $\mathrm{d}Q/\mathrm{d}x = -q$ for beams, $\nabla \cdot \underline{\sigma} + \vec{f} = 0$ for the continuum — express force equilibrium and hold for every material. **Kinematic relations** — $\chi = \varphi'$, $\underline{\varepsilon} = \frac{1}{2}(\nabla\vec{u} + \nabla\vec{u}^T)$ — are pure geometry. Neither knows whether the body is made of steel or of rubber. The missing link is the **constitutive relation** (material law), which relates the stress to the strain. For small deformations of most solids it is linear: **Hooke's law**.

Hooke's law in one dimension

Robert Hooke (1635–1703) observed that the extension of a spring is proportional to the force, $F = kx$. Consider instead a bar of initial length x_0 and cross-section A, clamped at one end and pulled with a force F at the other. It stretches to the length x,

$$F = k\,(x - x_0) = k\,x_0\,\frac{x - x_0}{x_0} = k\,x_0\,\varepsilon_{xx}.$$

The stiffness k is not a material property: a bar twice as long ($2x_0$) elongates twice as much under the same force, so its k is halved. Dividing by the cross-section gives the stress,

$$\sigma_{xx} = \frac{F}{A} = \frac{k\,x_0}{A}\,\varepsilon_{xx} \equiv E\,\varepsilon_{xx}.$$

The constant $E = k\,x_0/A$ is the same for both bars: **Young's modulus** is a property of the material alone. The geometry enters the stiffness of a structure only through A and x_0 — or, for a beam, through EA and EI.

Why are materials elastic?

The value of E spans more than six orders of magnitude, from soft rubber to diamond. The reason is that elasticity has two very different microscopic origins.

Energy elasticity: crystals, metals, ceramics, glasses

In a crystal such as silicon, atoms sit at the positions of a lattice and are held together by chemical bonds. Each bond acts like a small spring with stiffness k_b: stretching the crystal stretches the bonds and increases their energy. Applying the 1D argument above to a single bond of length a (the lattice spacing), with one bond per area a^2, gives

$$E \sim \frac{k_b}{a}.$$

With typical bond stiffnesses of tens of N/m and a of a few ångström, this gives $E \sim 100$ GPa. Ceramics, metals and glasses all have Young's moduli of this order. Because the elasticity comes from the energy of the bonds, it is called *energy elasticity*; it depends only weakly on temperature.

Entropy elasticity: polymers and rubbers

A polymer consists of long chain molecules (e.g. polyethylene, a chain of CH_2 units) that can rotate about their bonds. In an elastomer (rubber), the chains are linked to each other at a few points, the **cross-links**, forming a network. Between two cross-links a chain is a random coil that can adopt an enormous number of configurations. Stretching the network straightens the coils and reduces this number, i.e. it reduces the entropy. The restoring force comes from the thermal motion that drives the chains back to the more probable, coiled configurations. This *entropy elasticity* gives

$$E \approx 3\,\rho\,k_B T,$$

where ρ is the number density of network strands (chain segments between cross-links), k_B Boltzmann's constant and T the absolute temperature. At room temperature $k_B T \approx 26$ meV $\approx 4 \times 10^{-21}$ J; with $\rho \sim 10^{26}\,\mathrm{m}^{-3}$ we obtain $E \sim 1$ MPa, five orders of magnitude below a crystal. Unlike crystals, rubbers become *stiffer* when heated.

Material	Young's modulus E
Diamond	~1000 GPa
Steel	~210 GPa
Silicon	130–190 GPa (depends on direction)
Glass, aluminium	~70 GPa
Polyethylene, glassy polymers	~1 GPa
Rubber, PDMS	~1 MPa

In both cases the response is linear only for small strains. Hooke's law is the first, linear term of a Taylor expansion of the stress in the strain. In three dimensions, $\sigma_{xx} = E\varepsilon_{xx}$ is not the whole story: a bar pulled along x also contracts sideways, and shear stresses produce shear strains. The rest of this chapter builds the full 3D law.

In order to compute the deformation of an elastic body, we need a constitutive relation (material law) to close the equations of elastostatic equilibrium,

$$\nabla \cdot \underline{\sigma} = 0 \quad \text{and} \quad \underline{\varepsilon}(\vec{r}) = \frac{1}{2}\left(\nabla\vec{u} + (\nabla\vec{u})^T\right). \tag{1}$$

Since we will be working in linear elasticity, the constitutive equation is a linear relationship between $\underline{\sigma}$ and $\underline{\varepsilon}$. Rather than simply stating this relationship, we will derive it from fundamental physical principles by considering how isotropic materials respond to two fundamental types of deformation: *volumetric* (hydrostatic) deformation and *shear* (deviatoric) deformation.

Volumetric and deviatoric decomposition

Decomposing stress and strain

Any symmetric tensor can be decomposed into a *spherical* (or *hydrostatic*) part and a *deviatoric* part. For the stress tensor:

$$\underline{\sigma} = \sigma_h \underline{1} + \underline{s}$$

where $\sigma_h = \frac{1}{3}\mathrm{tr}\,\underline{\sigma} = \frac{1}{3}\sigma_{kk}$ is the *hydrostatic stress* (equal to the mean of the principal stresses), $\underline{1}$ is the identity tensor, and $\underline{s} = \underline{\sigma} - \sigma_h\underline{1}$ is the *deviatoric stress tensor*. The deviatoric stress has zero trace by construction: $\mathrm{tr}\,\underline{s} = 0$.

Similarly, for the strain tensor:

$$\underline{\varepsilon} = \varepsilon_h \underline{1} + \underline{e}$$

where $\varepsilon_h = \frac{1}{3}\mathrm{tr}\,\underline{\varepsilon} = \frac{1}{3}\varepsilon_{kk}$ is the *volumetric strain* (related to relative volume change by $\Delta V/V = 3\varepsilon_h = \varepsilon_{kk}$), and $\underline{e} = \underline{\varepsilon} - \varepsilon_h\underline{1}$ is the *deviatoric strain tensor*.

Physical meaning

The hydrostatic part represents uniform compression or expansion—a change in volume without change in shape. Imagine a cube being uniformly squeezed from all sides: it becomes a smaller cube but remains cubic.

The deviatoric part represents pure distortion—a change in shape without change in volume. Imagine shearing a cube into a parallelepiped: it changes shape but its volume remains constant.

This decomposition is fundamental because, for isotropic materials, these two types of deformation are *independent* and governed by *different* elastic constants.

The two fundamental elastic moduli

Bulk modulus: Response to volumetric deformation

Consider a material subjected to pure hydrostatic stress, i.e., uniform pressure from all directions:

$$\underline{\sigma} = -p\,\underline{1} \quad \text{(pressure)}$$

where p is the pressure (positive in compression). The hydrostatic stress is $\sigma_h = -p$.

For an isotropic material, this produces a purely volumetric strain (uniform contraction):

$$\underline{\varepsilon} = \varepsilon_h\,\underline{1}$$

The *bulk modulus* K (also called the compression modulus) relates these:

$$\sigma_h = K \cdot 3\varepsilon_h = K \cdot \varepsilon_{kk} \tag{2}$$

or equivalently, $-p = K \cdot (\Delta V/V)$.

The bulk modulus measures the material's resistance to uniform compression. It has units of pressure (Pa). Typical values are:

- Steel: $K \approx 160$ GPa
- Aluminum: $K \approx 76$ GPa
- Rubber: $K \approx 2$ GPa
- Water: $K \approx 2.2$ GPa

A material with large K is difficult to compress volumetrically.

Shear modulus: Response to shear deformation

Now consider a material subjected to pure shear stress. For example, a simple shear in the x-z plane:

$$\underline{\sigma} = \begin{pmatrix} 0 & 0 & \tau \\ 0 & 0 & 0 \\ \tau & 0 & 0 \end{pmatrix}$$

This stress is purely deviatoric (zero trace) and produces a purely deviatoric strain:

$$\underline{\varepsilon} = \begin{pmatrix} 0 & 0 & \varepsilon_{xz} \\ 0 & 0 & 0 \\ \varepsilon_{xz} & 0 & 0 \end{pmatrix}$$

The *shear modulus* G (also denoted μ and called Lamé's second parameter) relates these:

$$\tau = 2G\varepsilon_{xz} = G\gamma_{xz} \tag{3}$$

where $\gamma_{xz} = 2\varepsilon_{xz}$ is the engineering shear strain.

More generally, the deviatoric stress is proportional to the deviatoric strain:

$$\underline{s} = 2G\,\underline{e} \tag{4}$$

The shear modulus measures the material's resistance to shape change. Typical values are:

- Steel: $G \approx 80$ GPa
- Aluminum: $G \approx 26$ GPa
- Rubber: $G \approx 0.001$ GPa

A material with large G is stiff in shear (difficult to distort).

Building the full constitutive relation

Combining volumetric and shear response

For an isotropic linear elastic material, we now combine the two fundamental responses. The stress can be written as:

$$\underline{\sigma} = \sigma_h\,\underline{1} + \underline{s} = K\varepsilon_{kk}\,\underline{1} + 2G\,\underline{e}$$

Substituting $\underline{e} = \underline{\varepsilon} - \frac{1}{3}\varepsilon_{kk}\,\underline{1}$:

$$\underline{\sigma} = K\varepsilon_{kk}\,\underline{1} + 2G\left(\underline{\varepsilon} - \frac{1}{3}\varepsilon_{kk}\,\underline{1}\right)$$

Rearranging:

$$\underline{\sigma} = \left(K - \frac{2G}{3}\right) \varepsilon_{kk}\, \underline{1} + 2G\, \underline{\varepsilon}$$

In component form:

$$\sigma_{ij} = \left(K - \frac{2G}{3}\right) \delta_{ij}\varepsilon_{kk} + 2G\varepsilon_{ij} \tag{5}$$

This is Hooke's law expressed in terms of K and G.

Lamé's first parameter

The combination $K - \frac{2G}{3}$ appears naturally and is given a name: *Lamé's first parameter*

$$\lambda = K - \frac{2G}{3} \tag{6}$$

With this definition, Hooke's law takes the compact form:

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} \tag{7}$$

where we use $\mu = G$ (Lamé's second parameter, same as the shear modulus).

This is the *forward* constitutive relation: given the strain, compute the stress.

The inverse relation

To find strain from stress, we invert the relationship. The inverse of Hooke's law is:

$$\varepsilon_{ij} = \frac{1}{2G}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}] \tag{8}$$

where E and ν are Young's modulus and Poisson's ratio, which we derive next.

Young's modulus and Poisson's ratio

Uniaxial tension experiment

While K and G have clear physical interpretations (resistance to compression and shear), the most common laboratory experiment is the *uniaxial tension test*: a bar is stretched in one direction (say x) with no stress in the transverse directions.

The stress state is:

$$\sigma_{xx} = \sigma, \quad \sigma_{yy} = \sigma_{zz} = 0, \quad \text{all shear stresses} = 0$$

Using Eq. (8), the strain state is:

$$\varepsilon_{xx} = \frac{\sigma}{E}, \quad \varepsilon_{yy} = \varepsilon_{zz} = -\frac{\nu\sigma}{E}$$

This defines:

- **Young's modulus** E: the ratio of axial stress to axial strain in uniaxial tension

$$E = \frac{\sigma}{\varepsilon_{xx}}$$

- **Poisson's ratio** ν: the negative ratio of transverse strain to axial strain

$$\nu = -\frac{\varepsilon_{yy}}{\varepsilon_{xx}} = -\frac{\varepsilon_{zz}}{\varepsilon_{xx}}$$

Relating E, nu to K, G

We can derive the relationships by considering the uniaxial stress state. With $\sigma_{xx} = \sigma$ and all other stresses zero:

- Hydrostatic stress: $\sigma_h = \frac{\sigma}{3}$
- Volumetric strain: $\varepsilon_{kk} = \frac{\sigma_h}{K} = \frac{\sigma}{3K}$

For the axial strain, using the deviatoric part:

$$\varepsilon_{xx} = \varepsilon_h + e_{xx} = \frac{\varepsilon_{kk}}{3} + \frac{s_{xx}}{2G}$$

The deviatoric stress is $s_{xx} = \sigma_{xx} - \sigma_h = \sigma - \frac{\sigma}{3} = \frac{2\sigma}{3}$.

Therefore:

$$\varepsilon_{xx} = \frac{\sigma}{9K} + \frac{2\sigma}{6G} = \frac{\sigma}{9K} + \frac{\sigma}{3G} = \sigma\left(\frac{1}{9K} + \frac{1}{3G}\right)$$

Since $E = \sigma/\varepsilon_{xx}$:

$$\boxed{E = \frac{9KG}{3K + G}} \tag{9}$$

For the transverse strain:

$$\varepsilon_{yy} = \varepsilon_h + e_{yy} = \frac{\sigma}{9K} + \frac{s_{yy}}{2G}$$

The deviatoric stress is $s_{yy} = 0 - \frac{\sigma}{3} = -\frac{\sigma}{3}$.

Therefore:

$$\varepsilon_{yy} = \frac{\sigma}{9K} - \frac{\sigma}{6G}$$

The Poisson ratio is:

$$\nu = -\frac{\varepsilon_{yy}}{\varepsilon_{xx}} = \frac{-\frac{1}{9K} + \frac{1}{6G}}{\frac{1}{9K} + \frac{1}{3G}} = \frac{\frac{3K-2G}{18KG}}{\frac{3K+G}{9KG}} = \frac{3K - 2G}{2(3K + G)}$$

$$\boxed{\nu = \frac{3K - 2G}{2(3K + G)}} \tag{10}$$

Inverse relations

We can also express K and G in terms of E and ν:

$$K = \frac{E}{3(1 - 2\nu)} \tag{11}$$

$$G = \frac{E}{2(1 + \nu)} \tag{12}$$

And Lamé's first parameter:

$$\lambda = K - \frac{2G}{3} = \frac{E\nu}{(1 + \nu)(1 - 2\nu)} \tag{13}$$

! Poisson's ratio $\nu = 0.5$ means volume conservation

From Eq. (11), we see that:

$$K = \frac{E}{3(1 - 2\nu)}$$

As $\nu \to 0.5$, the bulk modulus $K \to \infty$. This means the material becomes *incompressible*: it cannot change its volume.
Let's verify this directly. The volumetric strain is:

$$\varepsilon_{kk} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

For uniaxial tension with $\varepsilon_{xx} = \sigma/E$ and $\varepsilon_{yy} = \varepsilon_{zz} = -\nu\sigma/E$:

$$\varepsilon_{kk} = \frac{\sigma}{E} - 2\frac{\nu\sigma}{E} = \frac{\sigma}{E}(1 - 2\nu)$$

When $\nu = 0.5$:

$$\varepsilon_{kk} = 0 \quad \Rightarrow \quad \frac{\Delta V}{V} = 0$$

The material stretches in the axial direction but contracts in the transverse directions in exactly the right proportion to keep the volume constant.
Physical examples:

- **Rubber** has $\nu \approx 0.5$ (nearly incompressible)
- **Water** and other liquids have $\nu = 0.5$ exactly (truly incompressible)
- **Metals** typically have $\nu \approx 0.3$ (compressible)

The physical bounds on Poisson's ratio are $-1 < \nu < 0.5$. The upper limit $\nu = 0.5$ corresponds to incompressibility. Materials with $\nu < 0$ (auxetic materials) expand transversely when stretched—these are typically architectured materials like certain foams.

General form of Hooke's law

The stiffness tensor

The most general form of the linear relationship between stress and strain is:

$$\underline{\sigma} = \underline{\underline{C}} : \underline{\varepsilon}, \quad \text{or} \quad \sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{14}$$

This is *Hooke's law*. The quantity $\underline{\underline{C}}$ is a fourth-order tensor of elastic constants called the *stiffness tensor*. For a general anisotropic material, it contains at most 21 independent elastic constants.

To see that there are only 21 independent coefficients, we use *Voigt notation*, which removes the symmetric entries from $\underline{\sigma}$ and $\underline{\varepsilon}$ and expresses them as 6-vectors:

$$\vec{\sigma} = (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{xz}, \sigma_{xy}) \tag{15}$$

and

$$\vec{\varepsilon} = (\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, 2\varepsilon_{yz}, 2\varepsilon_{xz}, 2\varepsilon_{xy}) \tag{16}$$

Then $\vec{\sigma} = \underline{C} \cdot \vec{\varepsilon}$ where $\underline{C}$ is a 6×6 symmetric matrix called the *stiffness matrix*. The symmetry reduces the 36 components to 21 independent constants.

Notation for shear components

The off-diagonal components of $\underline{\sigma}$ are often denoted by $\tau_{xy} \equiv \sigma_{xy}$, $\tau_{xz} \equiv \sigma_{xz}$, and $\tau_{yz} \equiv \sigma_{yz}$. The off-diagonal components of $\underline{\varepsilon}$ are often written as *engineering shear strains*: $\gamma_{xy} \equiv 2\varepsilon_{xy}$, $\gamma_{xz} \equiv 2\varepsilon_{xz}$, and $\gamma_{yz} \equiv 2\varepsilon_{yz}$.

Note: The γ's contain a factor of 2 but the τ's do not. This factor ensures that $\vec{\sigma} = \underline{C} \cdot \vec{\varepsilon}$ and $\underline{\sigma} = \underline{\underline{C}} : \underline{\varepsilon}$ are equivalent.

Stiffness matrix for isotropic materials

For isotropic materials, the stiffness matrix in Voigt notation is:

$$\underline{C} = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix}$$

This confirms that isotropic materials have only 2 independent elastic constants (λ and μ), compared to 21 for general anisotropic materials.

Elastic energy

We can also write down a *free energy functional* (often called hyperelastic energy density), which is quadratic in the strain $\underline{\varepsilon}$:

$$W = \frac{1}{2}\lambda\varepsilon_{ii}^2 + \mu\varepsilon_{ij}^2 \tag{17}$$

Using $\sigma_{ij} = \partial W/\partial\varepsilon_{ij}$ recovers Hooke's law (Eq. (7)).

In terms of K and G, the energy can be written as:

$$W = \frac{1}{2}K\varepsilon_{kk}^2 + G\, e_{ij}e_{ij} \tag{18}$$

This form clearly separates the volumetric energy (first term, proportional to K) from the shear/distortional energy (second term, proportional to G). For the energy to be positive definite (so that deformation costs energy), we require $K > 0$ and $G > 0$.

These constraints limit the Poisson ratio to the range $-1 < \nu < 1/2$.

Summary of elastic constants

The following table summarizes the relationships between pairs of elastic constants:

Given	E	ν	K	G	λ
K, G	$\frac{9KG}{3K+G}$	$\frac{3K-2G}{2(3K+G)}$	K	G	$K - \frac{2G}{3}$
E, ν	E	ν	$\frac{E}{3(1-2\nu)}$	$\frac{E}{2(1+\nu)}$	$\frac{E\nu}{(1+\nu)(1-2\nu)}$
λ, μ	$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\frac{\lambda}{2(\lambda+\mu)}$	$\lambda + \frac{2\mu}{3}$	μ	λ

Example: Uniaxial tension test

Tip

Consider a rectangular block of isotropic elastic material subjected to uniaxial tension. The block has Young's modulus $E = 200$ GPa and Poisson's ratio $\nu = 0.3$. An axial stress $\sigma_{xx} = 100$ MPa is applied in the x-direction with all other stresses zero: $\sigma_{yy} = \sigma_{zz} = \tau_{yz} = \tau_{xz} = \tau_{xy} = 0$.
Find:

- The axial strain ε_{xx}
- The transverse strains ε_{yy} and ε_{zz}
- The volumetric strain ε_h

Solution:
This is the simplest and most fundamental loading case in elasticity: a bar being stretched in one direction. We want to find how much the bar elongates in the loading direction and how much it contracts in the perpendicular directions.
We use the inverse constitutive relation (Eq. (8)) that gives strain in terms of stress:

$$\varepsilon_{ij} = \frac{1}{E}[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}]$$

This formula contains the trace of the stress tensor, σ_{kk}, which is the sum of the diagonal components. We calculate this first:

$$\sigma_{kk} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = 100 + 0 + 0 = 100 \text{ MPa}$$

Now we apply the formula to find each strain component. For the axial strain ε_{xx}, we set $i = j = x$. The Kronecker delta $\delta_{xx} = 1$, so:

$$\varepsilon_{xx} = \frac{1}{E}[(1+\nu)\sigma_{xx} - \nu\sigma_{kk}]$$

Substituting the values ($E = 200$ GPa $= 200 \times 10^3$ MPa, $\nu = 0.3$, $\sigma_{xx} = 100$ MPa, $\sigma_{kk} = 100$ MPa):

$$\varepsilon_{xx} = \frac{1}{200 \times 10^3}[(1 + 0.3)(100) - (0.3)(100)] = \frac{1}{200 \times 10^3}[130 - 30] = \frac{100}{200 \times 10^3}$$

$$= 0.5 \times 10^{-3} = 5 \times 10^{-4}$$

This means the bar elongates by 0.5 mm for every meter of original length (or equivalently, 0.05%).

For the transverse strain ε_{yy}, we set $i = j = y$. Again $\delta_{yy} = 1$, and now $\sigma_{yy} = 0$:

$$\varepsilon_{yy} = \frac{1}{E}[(1 + \nu)\sigma_{yy} - \nu\sigma_{kk}] = \frac{1}{200 \times 10^3}[(1.3)(0) - (0.3)(100)]$$

$$= \frac{1}{200 \times 10^3}[-30] = -0.15 \times 10^{-3} = -1.5 \times 10^{-4}$$

The negative sign indicates contraction: the bar gets thinner as it stretches. By symmetry of the loading and material (isotropic), the z-direction behaves the same as the y-direction: $\varepsilon_{zz} = -0.15 \times 10^{-3}$.

The volumetric strain ε_h (also called dilatation) is the sum of the normal strains:

$$\varepsilon_h = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = (0.5 - 0.15 - 0.15) \times 10^{-3} = 0.2 \times 10^{-3}$$

We can verify this using the alternative formula from the text. The hydrostatic stress is $\sigma_h = \sigma_{kk}/3 = 100/3$ MPa, and:

$$\varepsilon_h = \frac{1}{E}(1 - 2\nu)\sigma_{kk} = \frac{1}{200 \times 10^3}(1 - 0.6)(100) = \frac{40}{200 \times 10^3} = 0.2 \times 10^{-3}$$

This confirms our calculation. The positive volumetric strain means the material increases in volume when stretched—the elongation in the x-direction is not fully compensated by the contractions in the y and z directions. This is true for all materials with $\nu < 0.5$. An incompressible material ($\nu = 0.5$) would have $\varepsilon_h = 0$.

Finally, we can verify that the Poisson's ratio is recovered correctly. The ratio of transverse to axial strain should equal $-\nu$:

$$\frac{\varepsilon_{yy}}{\varepsilon_{xx}} = \frac{-0.15 \times 10^{-3}}{0.5 \times 10^{-3}} = -0.3 = -\nu \quad \checkmark$$

Plane problems

Plane problems are problems where the system has symmetry in a certain direction. We use the y-direction as the direction in which the plane conditions hold, meaning that all quantities are independent of y.

There are two main types of plane problems:

- **Plane strain** ($\varepsilon_{yy} = 0$): Applies to thick or constrained structures (e.g., dams, tunnels, long pipes)
- **Plane stress** ($\sigma_{yy} = 0$): Applies to thin structures (e.g., plates, membranes)

Both conditions reduce full 3D elasticity to 2D analysis in the x-z plane. The equilibrium condition in 2D becomes:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0 \tag{19}$$

$$\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \sigma_{xz}}{\partial x} = 0 \tag{20}$$

These two equations, together with Hooke's law (in plane strain or plane stress form) and the compatibility condition discussed below, form the complete set of equations for plane elasticity problems.

Plane stress

Plane stress greatly simplifies the 3D elasticity equations for thin structures like beams and membranes.

When does plane stress apply?

In full 3D elasticity, the stress tensor $\underline{\sigma}$ has six independent components. However, for structures that are thin in one direction (say, the y-direction), we can often assume that the stress components involving this direction vanish:

$$\sigma_{yy} = 0, \quad \tau_{xy} = 0, \quad \tau_{yz} = 0$$

This is called **plane stress** and is an excellent approximation when:

- The structure is thin (small extent in the y-direction compared to x and z)
- No forces are applied in the y-direction
- The surfaces perpendicular to y are traction-free

Under these conditions, the only non-zero stress components are:

$$\sigma_{xx}, \quad \sigma_{zz}, \quad \tau_{xz}$$

These stresses vary only in the x-z plane, hence the name "plane stress."

Simplified Hooke's law for plane stress

Under plane stress conditions, Hooke's law simplifies considerably. For an isotropic material with $\sigma_{yy} = 0$:

$$\varepsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{zz}) \tag{21}$$

$$\varepsilon_{zz} = \frac{1}{E}(\sigma_{zz} - \nu\sigma_{xx}) \tag{22}$$

$$\varepsilon_{yy} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{zz}) \tag{23}$$

$$\gamma_{xz} = \frac{\tau_{xz}}{G} = \frac{2(1+\nu)}{E}\tau_{xz} \tag{24}$$

Note that even though $\sigma_{yy} = 0$, the strain ε_{yy} is generally non-zero (the material contracts or expands freely in the y-direction due to the Poisson effect).

The inverse relations give stress from strain:

$$\sigma_{xx} = \frac{E}{1-\nu^2}(\varepsilon_{xx} + \nu\varepsilon_{zz}) \tag{25}$$

$$\sigma_{zz} = \frac{E}{1-\nu^2}(\nu\varepsilon_{xx} + \varepsilon_{zz}) \tag{26}$$

$$\tau_{xz} = G\gamma_{xz} \tag{27}$$

Application to beams

For a beam oriented along the x-axis with thickness in the z-direction and width in the y-direction, plane stress conditions apply when:

- The beam width (in y) is small compared to other dimensions
- The beam surfaces at $y = \pm t/2$ (where t is the width) are stress-free

This allows us to treat beam bending as a 2D problem in the x-z plane, as we did in Chapter 10.

Plane strain

The opposite limit is a body that is very *long* in the y-direction and held such that it cannot deform in y, e.g. a dam, a tunnel, a long pipe, or a thin film bonded to a thick substrate. Then

$$\varepsilon_{yy} = 0, \qquad \varepsilon_{xy} = 0, \qquad \varepsilon_{yz} = 0.$$

The out-of-plane strain vanishes, but the out-of-plane *stress* does not. From $\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right] = 0$,

$$\sigma_{yy} = \nu\left(\sigma_{xx} + \sigma_{zz}\right).$$

Inserting this into Hooke's law for the in-plane strains gives

$$\varepsilon_{xx} = \frac{1+\nu}{E}\left[(1-\nu)\,\sigma_{xx} - \nu\,\sigma_{zz}\right] \tag{28}$$

$$\varepsilon_{zz} = \frac{1+\nu}{E}\left[(1-\nu)\,\sigma_{zz} - \nu\,\sigma_{xx}\right] \tag{29}$$

$$\varepsilon_{xz} = \frac{1+\nu}{E}\,\sigma_{xz} = \frac{\sigma_{xz}}{2G}, \tag{30}$$

and the inverse relations

$$\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\,\varepsilon_{xx} + \nu\,\varepsilon_{zz}\right] \tag{31}$$

$$\sigma_{zz} = \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\,\varepsilon_{xx} + (1-\nu)\,\varepsilon_{zz}\right] \tag{32}$$

$$\sigma_{xz} = 2G\,\varepsilon_{xz}. \tag{33}$$

> **From plane stress to plane strain**
>
> The plane strain relations have the same form as the plane stress relations if we replace
>
> $$E \to \frac{E}{1-\nu^2}, \qquad \nu \to \frac{\nu}{1-\nu}.$$
>
> A solution obtained for plane stress can therefore be converted to plane strain (and vice versa). For $\nu \to 1/2$ the plane strain stiffness diverges: an incompressible material cannot be compressed in-plane if it cannot expand out of plane.

	Plane stress	Plane strain
Typical geometry	thin in y	long in y, constrained
Vanishing out-of-plane quantity	$\sigma_{yy} = 0$	$\varepsilon_{yy} = 0$
Non-zero out-of-plane quantity	$\varepsilon_{yy} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{zz})$	$\sigma_{yy} = \nu(\sigma_{xx} + \sigma_{zz})$

In both cases the in-plane problem is governed by the 2D equilibrium equations above, the 2D Hooke's law and the compatibility condition of Chapter 11.

Back to beams: deflection from elasticity

In Chapter 10 we derived the stresses inside a bent beam from equilibrium alone,

$$\sigma_{xx} = \frac{M(x)}{I_y} z, \qquad \tau_{xz} = \frac{Q(x)}{2I_y}\left(\frac{h^2}{4} - z^2\right), \qquad \sigma_{zz} = \frac{q(x)}{2I_y}\left(\frac{h^2}{4} - \frac{z^2}{3}\right) z.$$

With Hooke's law we can now compute the displacement and thereby the deformation of the beam — without assuming the Bernoulli hypothesis. Starting from Hooke's law in plane stress ($\sigma_{yy} = 0$),

$$\varepsilon_{xx} \equiv u_{x,x} = \sigma_{xx}/E - \nu\sigma_{zz}/E \tag{34}$$

$$2\varepsilon_{xz} \equiv u_{x,z} + u_{z,x} = 2(1+\nu)\tau_{xz}/E, \tag{35}$$

and taking the derivative of the second equation with respect to x, we obtain

$$u_{x,xz} + u_{z,xx} = 2(1+\nu)\tau_{xz,x}/E \tag{36}$$

and

$$u_{x,xz} + u_{z,xx} = \partial_z(u_{x,x}) + u_{z,xx} = u_{z,xx} + (\sigma_{xx,z} - \nu\sigma_{zz,z})/E. \tag{37}$$

Combining these two equations yields

$$u_{z,xx} = \left[2(1+\nu)\tau_{xz,x} - \sigma_{xx,z} + \nu\sigma_{zz,z}\right]/E, \tag{38}$$

where we now insert the three stresses from above. This gives

$$u_{z,xx} = \frac{1}{EI_y}\left[-\left(1 + \frac{\nu}{2}\right)\left(\frac{h^2}{4} - z^2\right) q(x) - M(x)\right], \tag{39}$$

which at the surface of the beam $w(x) = u_z(x, h/2)$ becomes

$$EI_y w_{,xx} = -M(x). \tag{40}$$

This equation is called the * Euler-Bernoulli beam equation*. By using $M_{,xx} = -q(x)$ we can write this as

$$(EI_y w_{,xx})_{,xx} = q(x), \tag{41}$$

in terms of the line load $q(x)$, or

$$EI_y w_{,xxxx} = q(x), \tag{42}$$

if EI_y does not depend on position x.

Chapter Summary

This chapter introduced the material law of linear elasticity:

- **Constitutive vs. balance equations**: equilibrium and kinematics hold for every material; Hooke's law closes the system
- **Hooke's law in 1D**: $\sigma_{xx} = E\varepsilon_{xx}$; Young's modulus E is a material property, unlike the stiffness k of a structure
- **Origin of elasticity**: energy elasticity of bonds in crystals ($E \sim 100$ GPa) and entropy elasticity of polymer networks ($E \approx 3\rho k_B T \sim 1$ MPa)
- **Volumetric and deviatoric decomposition**: the bulk modulus K resists volume changes, the shear modulus G resists shape changes
- **Elastic constants**: two independent constants for isotropic materials; E, ν, K, G, λ can be converted into each other
- **General Hooke's law**: $\underline{\sigma} = \underline{\underline{C}} : \underline{\varepsilon}$, with 21 independent constants in general (Voigt notation) and 2 for isotropic materials
- **Plane stress** ($\sigma_{yy} = 0$, thin bodies) and **plane strain** ($\varepsilon_{yy} = 0$, long constrained bodies) reduce the problem to 2D
- **Back to beams**: Hooke's law and the stresses of Chapter 10 give $EI_y w'' = -M$, the Euler–Bernoulli equation, without assuming plane sections