12. Virtual work in the continuum

Learning Objectives

After completing this chapter, you should be able to:

- **Write** the internal virtual work of a continuum, $\delta W_{\text{int}} = \int_V \underline{\sigma} : \delta\underline{\varepsilon}\, \mathrm{d}V$
- **Derive** the equilibrium equations $\nabla \cdot \underline{\sigma} + \vec{f} = 0$ and the traction boundary condition $\vec{t} = \underline{\sigma} \cdot \hat{n}$ from the principle of virtual work
- **Recognize** stress and strain as a conjugate pair and recover the beam case

We have now taken the first two steps for the continuum. Chapter 9 gave the **equilibrium** of an infinitesimal volume element,

$$\sigma_{ij,j} + f_i = 0 \quad \text{in } V, \qquad \sigma_{ij} n_j = t_{0,i} \quad \text{on the loaded surface,}$$

and Chapter 11 gave the **kinematics**, the strain tensor

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right).$$

In this chapter we take the third step, exactly as for the rigid body (Chapter 3) and for the beam (Chapter 6): we write down the principle of virtual work and derive the equilibrium equations from it once more. Along the way we learn that stress and strain form a **conjugate pair**, and we see that the beam theory of Chapter 6 is a special case of the continuum.

Virtual work of a continuum

Virtual displacement fields

A **virtual displacement** of a continuum is a vector field $\delta\vec{u}(\vec{r})$: an imagined, infinitesimal displacement of every material point that is compatible with the supports. We split the surface S of the body into the part S_u where displacements are prescribed (supports) and the part S_t where tractions $\vec{t}_0$ are prescribed (loads, or free surfaces with $\vec{t}_0 = 0$). Admissible virtual

displacements vanish on S_u. The **virtual strain** follows from the same kinematic relation as the real strain,

$$\delta\varepsilon_{ij} = \frac{1}{2}\left(\delta u_{i,j} + \delta u_{j,i}\right).$$

Internal virtual work

Consider a small cube of volume $\mathrm{d}V$. The stress σ_{xx} acts on the two faces normal to x; if the cube is stretched by the virtual strain $\delta\varepsilon_{xx}$, these faces move apart by $\delta\varepsilon_{xx}\,\mathrm{d}x$ and the work is $\sigma_{xx}\,\mathrm{d}y\,\mathrm{d}z \cdot \delta\varepsilon_{xx}\,\mathrm{d}x = \sigma_{xx}\,\delta\varepsilon_{xx}\,\mathrm{d}V$. The same holds for every component, so the internal virtual work per unit volume is $\sigma_{ij}\,\delta\varepsilon_{ij}$ (sum over i and j). For the whole body,

$$\boxed{\delta W_{\mathrm{int}} = \int_V \sigma_{ij}\,\delta\varepsilon_{ij}\,\mathrm{d}V = \int_V \underline{\sigma} : \delta\underline{\varepsilon}\,\mathrm{d}V.}$$

Stress and strain are a **conjugate pair**, just like N and ε, Q and γ, M and χ for the beam.

Because $\underline{\sigma}$ is symmetric, only the symmetric part of the displacement gradient contributes:

$$\sigma_{ij}\,\delta u_{i,j} = \sigma_{ij}\left(\delta\varepsilon_{ij} + \delta\omega_{ij}\right) = \sigma_{ij}\,\delta\varepsilon_{ij},$$

since the double contraction of a symmetric with an antisymmetric tensor vanishes. **Rotations do no internal work** — the continuum analogue of $\delta W_i = 0$ for the rigid body.

External virtual work

External loads are the body force $\vec{f}$ (per unit volume) and the surface traction $\vec{t}_0$ on S_t:

$$\delta W_{\mathrm{ext}} = \int_V f_i\,\delta u_i\,\mathrm{d}V + \int_{S_t} t_{0,i}\,\delta u_i\,\mathrm{d}A.$$

> **!** Principle of virtual work (continuum)
>
> A body is in equilibrium if and only if
>
> $$\int_V \sigma_{ij}\,\delta\varepsilon_{ij}\,\mathrm{d}V = \int_V f_i\,\delta u_i\,\mathrm{d}V + \int_{S_t} t_{0,i}\,\delta u_i\,\mathrm{d}A$$
>
> for every virtual displacement field $\delta\vec{u}$ that vanishes on S_u.

As for beams, the principle uses only equilibrium and kinematics and holds for any material.

Equilibrium from virtual work

We again move the derivative from the virtual displacement onto the stress, now with the product rule in 3D,

$$\sigma_{ij}\,\delta u_{i,j} = \left(\sigma_{ij}\,\delta u_i\right)_{,j} - \sigma_{ij,j}\,\delta u_i,$$

and convert the first term into a surface integral with the divergence theorem of Chapter 9:

$$\int_V \left(\sigma_{ij}\,\delta u_i\right)_{,j}\,\mathrm{d}V = \oint_S \sigma_{ij}\,n_j\,\delta u_i\,\mathrm{d}A.$$

This is the 3D version of integration by parts. The internal virtual work becomes

$$\delta W_{\text{int}} = \oint_S \sigma_{ij}\,n_j\,\delta u_i\,\mathrm{d}A - \int_V \sigma_{ij,j}\,\delta u_i\,\mathrm{d}V.$$

Inserting into the principle of virtual work, and using $\delta\vec{u} = 0$ on S_u,

$$-\int_V \left(\sigma_{ij,j} + f_i\right)\delta u_i\,\mathrm{d}V + \int_{S_t} \left(\sigma_{ij}n_j - t_{0,i}\right)\delta u_i\,\mathrm{d}A = 0.$$

Since $\delta\vec{u}$ is arbitrary inside V and on S_t, both brackets must vanish:

$$\boxed{\sigma_{ij,j} + f_i = 0 \quad \text{in } V, \qquad \sigma_{ij}\,n_j = t_{0,i} \quad \text{on } S_t.}$$

These are the equilibrium equation of the volume element and the traction boundary condition of Chapter 9. On every part of the surface and in every direction, we prescribe *either* the displacement *or* the traction — the same rule as for the ends of a beam.

Rigid virtual displacements

If we choose a rigid virtual displacement, $\delta\vec{u} = \delta\vec{u}_O + \delta\vec{\theta}\times\vec{r}$, the virtual strain vanishes and so does δW_{int}. The principle then states that the total external force and moment vanish — the equilibrium of the whole body of Chapters 1 and 3. Virtual work therefore contains the global equilibrium (rigid virtual displacements) and the local equilibrium (arbitrary virtual displacements) in a single statement.

Back to beams

We can now see where the beam theory comes from. Let the beam axis be x, with z pointing downward and the cross-section rotating by φ (Chapter 5). If the cross-section stays plane, a point at depth z moves by

$$u_x(x,z) = u(x) + z\,\varphi(x), \qquad u_z(x,z) = w(x).$$

(A positive, counterclockwise rotation moves points below the axis, $z > 0$, in the $+x$ direction.) The non-zero strains are

$$\varepsilon_{xx} = u_{x,x} = u' + z\,\varphi' = \varepsilon + z\,\chi, \qquad 2\varepsilon_{xz} = u_{x,z} + u_{z,x} = \varphi + w' = \gamma.$$

The 3D strain contains exactly the three beam strains of Chapter 5. Inserting the virtual counterparts into the internal virtual work and integrating over the cross-section A,

$$\delta W_{\text{int}} = \int_0^L \int_A (\sigma_{xx}\,\delta\varepsilon_{xx} + 2\sigma_{xz}\,\delta\varepsilon_{xz})\,\mathrm{d}A\,\mathrm{d}x = \int_0^L \Big(\underbrace{\int_A \sigma_{xx}\,\mathrm{d}A}_{N}\,\delta\varepsilon + \underbrace{\int_A z\,\sigma_{xx}\,\mathrm{d}A}_{M}\,\delta\chi + \underbrace{\int_A \sigma_{xz}\,\mathrm{d}A}_{Q}\,\delta\gamma \Big)\mathrm{d}x.$$

This is the internal virtual work of Chapter 6, and the internal forces appear automatically as the stress integrals of Chapters 9 and 10. The beam is a continuum with a restricted (plane-section) kinematics; its conjugate pairs are inherited from $\underline{\sigma}$–$\underline{\varepsilon}$.

Outlook: elastic energy

For a linear elastic material, $\underline{\sigma} = \underline{\underline{C}} : \underline{\varepsilon}$ (Chapter 13), and the work stored per unit volume is $\frac{1}{2}\underline{\sigma} : \underline{\varepsilon}$, the continuum version of $\frac{1}{2}M\chi$ that led to the elastic energy of a beam in Chapter 8. Virtual work and elastic energy are the basis of most numerical methods in solid mechanics, such as the finite element method.

Chapter Summary

This chapter applied the principle of virtual work to the continuum:

- **Virtual strain**: $\delta\varepsilon_{ij} = \frac{1}{2}(\delta u_{i,j} + \delta u_{j,i})$ for a virtual displacement field that vanishes on the supported surface S_u
- **Internal virtual work**: $\delta W_{\text{int}} = \int_V \underline{\sigma} : \delta\underline{\varepsilon}\,\mathrm{d}V$; stress and strain are a conjugate pair; rotations do no internal work
- **Principle of virtual work**: $\int_V \sigma_{ij}\delta\varepsilon_{ij}\,\mathrm{d}V = \int_V f_i\delta u_i\,\mathrm{d}V + \int_{S_t} t_{0,i}\delta u_i\,\mathrm{d}A$
- **Equilibrium**: the divergence theorem gives $\sigma_{ij,j} + f_i = 0$ in V and $\sigma_{ij}n_j = t_{0,i}$ on S_t
- **Rigid virtual displacements** give the global equilibrium of the body
- **Beams**: with plane-section kinematics $u_x = u + z\varphi$, $u_z = w$, the continuum virtual work reduces to $\int (N\delta\varepsilon + Q\delta\gamma + M\delta\chi)\,\mathrm{d}x$