10. Stresses in beams

Learning Objectives

After completing this chapter, you should be able to:

- **Derive** the beam equilibrium equations from 3D equilibrium with traction boundary conditions
- **Calculate** the bending stress $\sigma_{xx} = Mz/I_y$ and the moment of inertia $I_y = \int z^2 \, \mathrm{d}A$
- **Derive** the parabolic shear stress distribution τ_{xz} in a rectangular beam

Stresses inside beams

We now apply the equations above to a first problem: The stress inside of a beam. Consider a rectangular beam subject to bending. In the following derivation, we will assume that the stress fields do not vary in the y-direction (which is along the beam thickness). This means that all derivatives with respect to y will vanish. This is called a *plane* problem, in this case we will be working in what is called a *plane stress* situation.

The surface normal of the beam is oriented in z-direction and $z = 0$ is in the middle of the beam. Bending will give rise to internal stresses inside of the beam. We will require that these stresses comply with the external shear force $Q(x)$ and bending moment $M(x)$ in the *weak* or integral sense,

$$Q(x) = \int_A \mathrm{d}y\mathrm{d}z \, \tau_{xz}(x, z) \tag{1}$$

$$M(x) = \int_A \mathrm{d}y\mathrm{d}z \, z\sigma_{xx}(x, z). \tag{2}$$

Because of the plane state, integration in y-direction will only yield a constant factor, the width t of the beam.

We now simply take the derivatives of Equation 1 and Equation 2 with respect to the beam axis x. Taking the derivative of $Q(x)$ yields

$$\frac{\mathrm{d}Q}{\mathrm{d}x} = \int_A \mathrm{d}y\mathrm{d}z\, \frac{\mathrm{d}\tau_{xz}}{\mathrm{d}x} = -\int_A \mathrm{d}y\mathrm{d}z\, \frac{\mathrm{d}\sigma_{zz}}{\mathrm{d}z} = -t\left[\sigma_{zz}(x, h/2) - \sigma_{zz}(x, -h/2)\right].$$

We call the quantity $q(x) \equiv t\left[\sigma_{zz}(x, h/2) - \sigma_{zz}(x, -h/2)\right]$ the line load of the beam. Next, we take the derivative of $M(x)$,

$$\frac{\mathrm{d}M}{\mathrm{d}x} = \int_A \mathrm{d}y\mathrm{d}z\, z\frac{\mathrm{d}\sigma_{xx}}{\mathrm{d}x} = -\int_A \mathrm{d}y\mathrm{d}z\, z\frac{\mathrm{d}\tau_{xz}}{\mathrm{d}z} = -t\left[z\tau_{xz}\right]_{-h/2}^{h/2} + \int_A \mathrm{d}y\mathrm{d}z\, \tau_{xz}.$$

The first term on the right hand side vanishes because the surfaces are traction free, $\tau_{xz} = 0$ at $z = h/2$ and $z = -h/2$. The second term we recognize as the shear force $Q(x)$. We have therefore derived the equations

$$\frac{\mathrm{d}Q}{\mathrm{d}x} = -q(x) \quad \text{and} \quad \frac{\mathrm{d}M}{\mathrm{d}x} = Q(x),$$

which are the equation for the equilibrium of internal forces in beams.

i Applying traction boundary conditions to the beam

We have discussed traction boundary conditions in the previous chapter. Here we apply these concepts to the beam. The top and bottom surfaces of the beam (at $z = \pm h/2$) have normals pointing in the z-direction. As discussed in the previous chapter, the traction boundary condition fixes three stress components at each surface: σ_{zz}, σ_{xz}, and σ_{yz}. For a beam loaded only by a distributed line load $q(x)$ on its top surface:

- The shear stresses vanish at both surfaces: $\tau_{xz} = 0$ at $z = \pm h/2$ (traction-free in the tangential direction)
- The normal stress σ_{zz} at the surfaces relates to the applied load

Since the stress tensor is a field that varies throughout the body, these surface values are boundary conditions that constrain the solution in the interior.

To arrive at a closed-form solution for the stresses inside the beam, we now *assume* that the stress is a linear function of the position z perpendicular to the beam axis,

$$\sigma_{xx}(x, z) = C(x)z, \tag{3}$$

with some as-of-yet unknown function for the coefficient $C(x)$. In what follows we show that this is a good assumption, i.e., we can fulfill force and moment equilibrium and the resulting strains fulfill the compatibility conditions.

It is straightforward to compute the bending moment,

$$M(x) = \int_A \mathrm{d}y\mathrm{d}z\, \sigma_{xx}(x, z)z = C(x)\int_A \mathrm{d}y\mathrm{d}z\, z^2 = C(x)I_y, \tag{4}$$

where A is the cross-section of the beam and

$$I_y = \int_A \mathrm{d}y\mathrm{d}z\, z^2$$

is the *axial moment of inertia*. This is a number that characterizes the shape of the beam. For a rectangular beam of height h and width t, $I_y = h^3 t/12$. The moment of inertia is a geometric factor and depends on the shape of the cross-section of the beam.

We can rewrite Equation 3 as

$$\sigma_{xx}(x,z) = \frac{M(x)}{I_y} z. \tag{5}$$

This equation is knows as Navier's bending formula, or just flexure formula. Note that an additional axial force $N(x)$ will simple be an additive contribution to Equation 5, $\sigma_{xx}(x,z) = M(x)/I_y z + N(x)/A$.

We can now use the condition for static equilibrium to compute the full stress tensor $\underline{\sigma}$. From $\partial_x \sigma_{xx} + \partial_z \tau_{xz} = 0$ we obtain

$$\frac{\partial \tau_{xz}}{\partial z} = -\frac{z}{I_y}\frac{\mathrm{d}M}{\mathrm{d}x} = -\frac{z}{I_y} Q(x).$$

We can integrate this across the height h of the beam, keeping in mind that $\tau_{xz} = 0$ at a traction-free surface, to yield

$$\tau_{xz}(x,z) = \frac{Q(x)}{2I_y}\left(\frac{h^2}{4} - z^2\right). \tag{6}$$

This equation is known as the Jourawski's formula or shear stress formula. Next, we use $\partial_z \sigma_{zz} + \partial_x \tau_{xz} = 0$ to obtain

$$\frac{\partial \sigma_{zz}}{\partial z} = -\frac{1}{2I_y}\frac{\mathrm{d}Q}{\mathrm{d}x}\left(\frac{h^2}{4} - z^2\right) = \frac{q(x)}{2I_y}\left(\frac{h^2}{4} - z^2\right).$$

We need to integrate this equation again, but now $\sigma_{zz} \neq 0$ at the surface since the beam is subject to a line load $q(x)$. Rather, we need the condition that the loads on top and bottom surface of the beam balance, $\sigma_{zz}(x, h/2) = -\sigma_{zz}(x, -h/2)$. This gives

$$\sigma_{zz}(x,z) = \frac{q(x)}{2I_y}\left(\frac{h^2}{4} - \frac{z^2}{3}\right) z \tag{7}$$

with

$$\sigma_{zz}(x, h/2) = \frac{q(x)h^3}{24 I_y} = \frac{q(x)}{2t} \tag{8}$$

where t is the width of the beam.

Connection to Euler-Bernoulli beam theory

In this section, we derived the stress tensor inside a beam using the plane stress approximation of Linear Elasticity. This is a generalization that includes the the *Euler-Bernoulli beam theory* we have seen in previous chapters. However, the starting point of this theory is typically not Equation 3, but the assumption that each cross section will remain plane and undergo small rotations during deformation. These are sometimes called the *Bernoulli assumptions* and are valid if the beam is slender ($h << L$). The Euler-Bernoulli theory implies that the strain $\varepsilon_{xx} \propto z$ rather than the stress. It follows that $\varepsilon_{xx} = \frac{\sigma_{xx}}{E}$, but this of course ignores the other component of the stress tensor σ_{zz}. In fact, according to plain stress conditions in Linear Elasticity $\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu\sigma_{zz}}{E}$. However, we have seen that $\sigma_{zz} \propto qh^2$, whereas $\sigma_{xx} \propto M \approx qL^2$. Thus, it is reasonable for slender beams ($h << L$) to consider $\sigma_{zz} \approx 0$, as the axial stress is predominant. It is also interesting to notice that σ_{zz} vanishes anyway if there is no distributed load. Therefore, the critical stresses in slender beams are always caused by the bending moment, making the Euler-Bernoulli theory a powerful and valid tool.

Chapter Summary

This chapter connected the continuum description to beam theory:

- **Beam equilibrium from 3D**: $\mathrm{d}Q/\mathrm{d}x = -q$ and $\mathrm{d}M/\mathrm{d}x = Q$ follow from $\nabla \cdot \underline{\sigma} = 0$ with traction boundary conditions
- **Bending stress**: $\sigma_{xx} = Mz/I_y$ varies linearly through the cross-section
- **Shear stress**: $\tau_{xz} = Q(h^2/4 - z^2)/(2I_y)$ is parabolic through the thickness
- **Moment of inertia**: $I_y = \int z^2 \,\mathrm{d}A$ ($I_y = h^3 t/12$ for a rectangular cross-section)