6. Virtual work for beams

Learning Objectives

After completing this chapter, you should be able to:

- **Write** the internal virtual work of a beam as $\delta W_{\text{int}} = \int (N\,\delta\varepsilon + Q\,\delta\gamma + M\,\delta\chi)\,\mathrm{d}x$
- **Derive** the beam equilibrium equations and the static boundary conditions from the principle of virtual work
- **Identify** the conjugate pairs N–ε, Q–γ, M–χ
- **Use** virtual displacements to compute reactions of statically determinate beams

We now have two descriptions of a beam. Chapter 4 gave its **equilibrium**: the internal forces N, Q, M satisfy

$$\frac{\mathrm{d}N}{\mathrm{d}x} = -p(x), \qquad \frac{\mathrm{d}Q}{\mathrm{d}x} = -q(x), \qquad \frac{\mathrm{d}M}{\mathrm{d}x} = Q(x),$$

where $p(x)$ is an axial line load (we add it here for completeness) and $q(x)$ the transverse line load. Chapter 5 gave its **kinematics**: the displacements u, w, φ and the strains

$$\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x}, \qquad \gamma = \frac{\mathrm{d}w}{\mathrm{d}x} + \varphi, \qquad \chi = \frac{\mathrm{d}\varphi}{\mathrm{d}x}.$$

In this chapter we connect the two with the principle of virtual work, exactly as we did for the rigid body in Chapter 3. The result is a second derivation of the equilibrium equations, together with the boundary conditions, and a precise statement of which internal force belongs to which strain.

Virtual work of a deformable beam

Virtual displacements and virtual strains

A **virtual displacement** of the beam is a set of three functions $\delta u(x)$, $\delta w(x)$, $\delta\varphi(x)$: an imagined, infinitesimal displacement field that is compatible with the supports. For example, at a clamped end $\delta u = \delta w = \delta\varphi = 0$, and at a hinge $\delta u = \delta w = 0$ while $\delta\varphi$ is free. Unlike in

Chapter 3, the virtual displacement does not have to be rigid. It produces **virtual strains** via the same kinematic relations as the real displacements,

$$\delta\varepsilon = \frac{\mathrm{d}\,\delta u}{\mathrm{d}x}, \qquad \delta\gamma = \frac{\mathrm{d}\,\delta w}{\mathrm{d}x} + \delta\varphi, \qquad \delta\chi = \frac{\mathrm{d}\,\delta\varphi}{\mathrm{d}x}.$$

Internal virtual work

In a rigid body the internal forces do no work. In a deformable body they do: consider a beam element of length $\mathrm{d}x$.

- If the element is stretched by $\delta\varepsilon\,\mathrm{d}x$, the normal force N does the work $N\,\delta\varepsilon\,\mathrm{d}x$.
- If the element is sheared by the angle $\delta\gamma$, its two faces move transversally relative to each other by $\delta\gamma\,\mathrm{d}x$ and the shear force does the work $Q\,\delta\gamma\,\mathrm{d}x$.
- If the two faces rotate relative to each other by $\delta\varphi(x+\mathrm{d}x) - \delta\varphi(x) = \delta\chi\,\mathrm{d}x$, the bending moment does the work $M\,\delta\chi\,\mathrm{d}x$.

Summing over all elements, the **internal virtual work** (the work absorbed by the deformation of the beam) is

$$\boxed{\delta W_{\text{int}} = \int_0^L (N\,\delta\varepsilon + Q\,\delta\gamma + M\,\delta\chi)\,\mathrm{d}x.}$$

Each internal force multiplies exactly one strain. We call N and ε, Q and γ, M and χ **conjugate pairs**.

External virtual work

The external loads do work on the displacements of the points where they act. With line loads $p(x)$ and $q(x)$, and with point forces and couples at the ends of the beam,

$$\delta W_{\text{ext}} = \int_0^L (p\,\delta u + q\,\delta w)\,\mathrm{d}x + (\text{work of the end loads}).$$

Point loads inside the beam are included with the Dirac delta of Chapter 4.

Principle of virtual work for beams

> **!** Principle of virtual work (beams)
>
> A beam is in equilibrium if and only if
>
> $$\delta W_{\text{int}} = \delta W_{\text{ext}}$$
>
> for every virtual displacement δu, δw, $\delta\varphi$ that is compatible with the supports.

For a rigid body $\delta W_{\text{int}} = 0$ and we recover Chapter 3. Note that the principle uses only statics and kinematics. It does not depend on the material: it holds for elastic, plastic or any other beam.

Equilibrium from virtual work

We now show that the principle contains the equilibrium equations of Chapter 4. The internal virtual work contains derivatives of the virtual displacements. We move them onto the internal forces by integration by parts, $\int_0^L f g'\,\mathrm{d}x = [fg]_0^L - \int_0^L f' g\,\mathrm{d}x$:

$$\int_0^L N\,\delta u'\,\mathrm{d}x = \big[N\,\delta u\big]_0^L - \int_0^L N'\,\delta u\,\mathrm{d}x,$$

$$\int_0^L Q\,(\delta w' + \delta\varphi)\,\mathrm{d}x = \big[Q\,\delta w\big]_0^L - \int_0^L Q'\,\delta w\,\mathrm{d}x + \int_0^L Q\,\delta\varphi\,\mathrm{d}x,$$

$$\int_0^L M\,\delta\varphi'\,\mathrm{d}x = \big[M\,\delta\varphi\big]_0^L - \int_0^L M'\,\delta\varphi\,\mathrm{d}x.$$

Adding up,

$$\delta W_{\text{int}} = \big[N\,\delta u + Q\,\delta w + M\,\delta\varphi\big]_0^L - \int_0^L \Big(N'\,\delta u + Q'\,\delta w + (M' - Q)\,\delta\varphi\Big)\mathrm{d}x.$$

Setting this equal to δW_{ext} and collecting the terms inside the beam gives

$$\int_0^L \Big[(N' + p)\,\delta u + (Q' + q)\,\delta w + (M' - Q)\,\delta\varphi\Big]\mathrm{d}x + (\text{boundary terms}) = 0.$$

This must hold for *every* choice of $\delta u(x)$, $\delta w(x)$ and $\delta\varphi(x)$. We can, for instance, choose a virtual displacement that is non-zero only in a tiny neighbourhood of some point x. The integral can then only vanish if each bracket vanishes everywhere:

$$\boxed{\frac{\mathrm{d}N}{\mathrm{d}x} = -p, \qquad \frac{\mathrm{d}Q}{\mathrm{d}x} = -q, \qquad \frac{\mathrm{d}M}{\mathrm{d}x} = Q.}$$

These are exactly the equilibrium equations of the infinitesimal beam element from Chapter 4 — now obtained without drawing a single free body diagram.

Boundary conditions

The boundary terms $\left[N\,\delta u + Q\,\delta w + M\,\delta\varphi\right]_0^L$ tell us what happens at the ends. Consider the right end $x = L$, loaded by an axial force P, a transverse force F (positive in $+z$, i.e. downward) and a couple C (positive in the sense of φ). The end loads contribute $P\,\delta u(L) + F\,\delta w(L) + C\,\delta\varphi(L)$ to δW_{ext}, so the principle requires

$$\big(N(L) - P\big)\,\delta u(L) + \big(Q(L) - F\big)\,\delta w(L) + \big(M(L) - C\big)\,\delta\varphi(L) = 0.$$

For each of the three pairs there are two possibilities:

- the **displacement is prescribed** by a support (e.g. $w(L) = 0$): then $\delta w(L) = 0$, the term drops out, and the conjugate force (the reaction) is not known in advance;
- the **displacement is free**: then $\delta w(L)$ is arbitrary and the conjugate force must equal the applied load, $Q(L) = F$.

At every end and for every pair, we prescribe *either* the displacement (kinematic boundary condition) *or* the conjugate force (static boundary condition) — never both, and never neither:

Pair	Kinematic condition	Static condition
(u, N)	u prescribed	$N = P$
(w, Q)	w prescribed	$Q = F$
(φ, M)	φ prescribed	$M = C$

A clamped end prescribes u, w, φ; a hinge prescribes u, w and leaves $M = 0$; a free end has N, Q, M equal to the applied loads (zero if unloaded). This explains the boundary conditions we used in Chapter 4 and that we will use again for the Euler–Bernoulli beam in Chapter 7. At the left end $x = 0$ the same reasoning applies with a minus sign from the lower integration limit, because there the outward direction of the beam is $-x$.

Using virtual displacements in practice

Reactions of statically determinate beams

If we choose a virtual displacement that moves (parts of) the beam *rigidly*, all virtual strains vanish and $\delta W_{\text{int}} = 0$. To compute a reaction, we remove the corresponding support and let the beam move as a mechanism, exactly as in Chapter 3.

Example. A simply supported beam of length L carries a uniform load q_0. To find the reaction R_B (upward) of the roller at $x = L$, we remove the roller and rotate the beam rigidly about the hinge at A: $\delta w(x) = x\,\delta\alpha$. Then

$$\delta W_{\text{ext}} = \int_0^L q_0\, x\,\delta\alpha\,\mathrm{d}x - R_B\,L\,\delta\alpha = \left(\frac{q_0 L^2}{2} - R_B L\right)\delta\alpha = 0 \quad \Rightarrow \quad R_B = \frac{q_0 L}{2}.$$

Internal forces: releasing a moment with a hinge

The same idea gives internal forces. To compute the bending moment at a point $x = c$, we insert a hinge at c. The beam becomes a mechanism of two rigid parts that can rotate relative to each other at the hinge. A relative rotation $\Delta\varphi$ at a single point corresponds to a curvature concentrated at that point, $\delta\chi = \Delta\varphi\,\delta(x - c)$, so

$$\delta W_{\text{int}} = \int_0^L M\,\delta\chi\,\mathrm{d}x = M(c)\,\Delta\varphi.$$

Example. For the same simply supported beam, we insert a hinge at midspan, $c = L/2$, and choose the virtual displacement

$$\delta w(x) = \begin{cases} x\,\delta\alpha & 0 \le x \le L/2 \\ (L - x)\,\delta\alpha & L/2 \le x \le L. \end{cases}$$

Both halves stay rigid and Bernoulli's condition $\delta\varphi = -\delta w'$ holds within each half, so $\delta\varphi = -\delta\alpha$ on the left half and $\delta\varphi = +\delta\alpha$ on the right half. The relative rotation at the hinge is $\Delta\varphi = 2\,\delta\alpha$. The external work is the load times the area under the triangle $\delta w(x)$:

$$\delta W_{\text{ext}} = q_0 \cdot \frac{1}{2}\,L\,\frac{L}{2}\,\delta\alpha = \frac{q_0 L^2}{4}\,\delta\alpha.$$

From $M(L/2) \cdot 2\,\delta\alpha = q_0 L^2 \delta\alpha/4$ we get

$$M\left(\tfrac{L}{2}\right) = \frac{q_0 L^2}{8},$$

the well-known maximum moment of a simply supported beam, with a positive sign (bottom in tension).

> Rigid body, beam, continuum
>
> The principle of virtual work has the same structure at every level of the course:

Model	Forces	Conjugate kinematic quantities	Internal virtual work
Rigid body (Ch. 3)	$\vec{F}$, $\vec{M}$	$\delta\vec{u}_O$, $\delta\vec{\theta}$	0
Beam (this chapter)	N, Q, M	$\delta\varepsilon$, $\delta\gamma$, $\delta\chi$	$\int(N\delta\varepsilon + Q\delta\gamma + M\delta\chi)\,\mathrm{d}x$
Continuum (Ch. 12)	$\underline{\sigma}$	$\delta\underline{\varepsilon}$	$\int_V \underline{\sigma} : \delta\underline{\varepsilon}\,\mathrm{d}V$

Chapter Summary

This chapter derived the beam equations from the principle of virtual work:

- **Virtual strains**: $\delta\varepsilon = \delta u'$, $\delta\gamma = \delta w' + \delta\varphi$, $\delta\chi = \delta\varphi'$
- **Internal virtual work**: $\delta W_{\text{int}} = \int_0^L (N\,\delta\varepsilon + Q\,\delta\gamma + M\,\delta\chi)\,\mathrm{d}x$; the conjugate pairs are N–ε, Q–γ, M–χ
- **Principle of virtual work**: equilibrium $\Leftrightarrow \delta W_{\text{int}} = \delta W_{\text{ext}}$ for all admissible virtual displacements; it is independent of the material
- **Equilibrium**: integration by parts gives $N' = -p$, $Q' = -q$, $M' = Q$
- **Boundary conditions**: at each end, either the displacement or its conjugate force is prescribed
- **Practical use**: rigid virtual displacements of a released structure give reactions; a virtual hinge gives the bending moment

The next chapter adds the last missing ingredient, the material law that links each internal force to its conjugate strain.